Solutions of Assignment # 7.

Problem 1. Let $\ell \in \mathbb{R}$, $c \in \mathbb{R}$, $c > 0$. Show that the following two statements are equivalent.

a. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad |x_n - \ell| < \varepsilon$;

b. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad |x_n - \ell| < c\varepsilon$.

Solution. To avoid misleading notation we rewrite the first statement as

A. $\forall \delta > 0 \ \exists m \in \mathbb{N} \ \forall n \geq m \quad |x_n - \ell| < \delta$.

Clearly, Statement **A** is exactly the same as Statement **a.**

Par 1. A $\implies$ b. We assume that Statement **A** is true.

Fix an arbitrary $\varepsilon > 0$. Apply Statement **A** with $\delta = c\varepsilon > 0$. There exist $m \in \mathbb{N}$ such that $\forall n \geq m \quad |x_n - \ell| < \delta = c\varepsilon$. Choose $N = m$. Then $\forall n \geq N \quad |x_n - \ell| < c\varepsilon$. We proved **b.**

Par 2. b $\implies$ A. We assume that Statement **b** is true.

Fix an arbitrary $\delta > 0$. Apply Statement **b** with $\varepsilon = \delta/c > 0$. There exists $N \in \mathbb{N}$ such that $\forall n \geq N \quad |x_n - \ell| < c\varepsilon = \delta$. Choose $m = N$. Then $\forall n \geq N \quad |x_n - \ell| < \delta$. We proved **A.** □

Remark. Recall that to prove a statement like $\forall x \ \exists y$... one has to prove an existence of a "good" function $y(x)$. Note that Statement **A** provides the existence of a "good" function $m = m(\delta)$, while Statement **b** provides the existence of a "good" function $N = N(\varepsilon)$. In Part 1 of our proof we used the existence of $m(\delta)$ and defined $N(\varepsilon) = m(c\varepsilon)$. In Part 2, we used the existence of $N(\varepsilon)$ and defined $m(\delta) = N(\delta/c)$.

Problem 2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence convergent to 0. Let $\{y_n\}_{n=1}^{\infty}$ be a bounded sequence. Show that

$$\lim_{n \to \infty} (x_n y_n) = 0.$$

Solution.

Since $\{y_n\}_{n=1}^{\infty}$ is a bounded sequence, there exists $M > 0$ such that for every $n \in \mathbb{N} \quad |y_n| \leq M$. Now fix $\varepsilon > 0$. Since $x_n \to 0$, applying the definition of the limit (see Statements **a** and **A** in Problem 1 above) with $\delta = \varepsilon/M$, we obtain that there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ one has $|x_n| < \delta = \varepsilon/M$. Therefore, for $n \geq N$ we have $|x_n y_n| \leq |x_n|\,|y_n| < \delta M = \varepsilon$. So we proved

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad |x_n y_n - 0| < \varepsilon, \qquad \text{that is} \qquad \lim_{n \to \infty} (x_n y_n) = 0.$$

□

Problem 3. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence convergent to 0. Is it true that for every sequence $\{y_n\}_{n=1}^{\infty}$ one has

$$\lim_{n \to \infty} (x_n y_n) = 0$$

Answer and solution. No. For every n define $x_n = 1/n$ and $y_n = n$. Then $x_n \to 0$ and $\{y_n\}_{n=1}^{\infty}$ is divergent (as an unbounded sequence). However, $x_n y_n \to 1$ as a constant sequence. □

Problem 4. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence convergent to ℓ.
a. Prove that

$$\lim_{n\to\infty} x_n^2 = \ell^2.$$

b. Assuming that for every $n \in \mathbb{N}$ $\quad a_n \geq 0$ and $\ell \geq 0$, prove that

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\ell}.$$

Solution.
a. Way 1. By a theorem proved in the class we have

$$\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} x_n x_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} x_n = \ell^2.$$

a. Way 2. First assume $\ell \neq 0$. Note $x_n^2 - \ell^2 = (x_n - \ell)(x_n + \ell)$. Applying the definition of the limit with $\varepsilon = |\ell| > 0$ we observe that there exists N_1 such that for every $n \geq N_1$ one has $|x_n - \ell| < |\ell|$. It implies

$$\forall n \geq N_1 \quad |x_n| \leq |x_n - \ell| + |\ell| < 2|\ell|.$$

Now fix an arbitrary $\varepsilon > 0$. Apply the definition of the limit with $\varepsilon/|2\ell| > 0$. There exists N_2 such that for every $n \geq N_2$ one has $|x_n - \ell| < \varepsilon/|2\ell|$. Choose $N = \max\{N_1, N_2\}$. Then for every $n \geq N$ one has

$$|x_n^2 - \ell^2| = |x_n - \ell|\,|x_n + \ell| < \frac{\varepsilon}{|2\ell|}\, 2|\ell| = \varepsilon.$$

It proves the result.

The case $\ell = 0$ is simple (**exer.**).

b. Case 1 $\ell > 0$. First, using $x_n \geq 0$ and $\ell > 0$, we observe

$$\left|\sqrt{x_n} - \sqrt{\ell}\right| = \left|\frac{(\sqrt{x_n} - \sqrt{\ell})(\sqrt{x_n} + \sqrt{\ell})}{\sqrt{x_n} + \sqrt{\ell}}\right| = \frac{|x_n - \ell|}{\sqrt{x_n} + \sqrt{\ell}} \leq \frac{|x_n - \ell|}{\sqrt{\ell}}.$$

Now fix an arbitrary $\varepsilon > 0$. Apply the definition of the limit with $\sqrt{\ell}\,\varepsilon > 0$. There exists N such that for every $n \geq N$ one has $|x_n - \ell| < \sqrt{\ell}\,\varepsilon$. It implies that for every $n \geq N$ one has

$$|\sqrt{x_n} - \sqrt{\ell}| \leq \frac{|x_n - \ell|}{\sqrt{\ell}} < \frac{\sqrt{\ell}\,\varepsilon}{\sqrt{\ell}} = \varepsilon.$$

It proves the result.

b. Case 2 $\ell = 0$. We have $x_n \to 0$. Fix an arbitrary $\varepsilon > 0$ and apply the definition of the limit with $\varepsilon^2 > 0$. There exists N such that for every $n \geq N$ one has $|x_n| < \varepsilon^2$. It implies that for every $n \geq N$ one has $|\sqrt{x_n}| < \varepsilon$. So we proved that for every $\varepsilon > 0$ there exists N such that for every $n \geq N$ one has $|\sqrt{x_n}| < \varepsilon$. It proves the result. $\square$

Problem 5. Find limits of the following sequences (you can use facts proved in the class).

a. $\left\{\dfrac{2n^2 + n + \sqrt{n} - 3}{n^2 - 5n + 7}\right\}_{n=1}^{\infty}$, $\qquad$ **b.** $\left\{\dfrac{1 + 2 + 3 + \ldots + n}{n^2}\right\}_{n=1}^{\infty}$.

Solution.

a. We first note that for every $c \in \mathbb{R}$ and every $p > 0$ one has

$$\lim_{n\to\infty} \frac{c}{n^p} = 0.$$

Indeed, for every $\varepsilon > 0$ take $N > (1/\varepsilon)^{1/p}$. Then for every $n \geq N$ we have

$$\left| \frac{1}{n^p} - 0 \right| \leq \frac{1}{N^p} < \varepsilon.$$

It shows that $\lim_{n\to\infty} \frac{1}{n^p} = 0$, which by a Theorem in the class implies

$$\lim_{n\to\infty} \frac{c}{n^p} = c \lim_{n\to\infty} \frac{1}{n^p} = 0.$$

Now note

$$\frac{2n^2 + n + \sqrt{n} - 3}{n^2 - 5n + 7} = \frac{2 + 1/n + 1/n^{3/2} - 3/n^2}{1 - 5/n + 7/n^2}$$

and by above

$$\lim_{n\to\infty} 1/n = \lim_{n\to\infty} 1/n^{3/2} = \lim_{n\to\infty} 3/n^2 = \lim_{n\to\infty} 5/n = \lim_{n\to\infty} 7/n^2 = 0.$$

Using Theorems from the class we obtain,

$$\lim_{n\to\infty} \frac{2n^2 + n + \sqrt{n} - 3}{n^2 - 5n + 7} = \frac{\lim_{n\to\infty}(2 + 1/n + 1/n^{3/2} - 3/n^2)}{\lim_{n\to\infty}(1 - 5/n + 7/n^2)} = \frac{2}{1} = 2.$$

b. In class we proved $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. Therefore,

$$\lim_{n\to\infty} \frac{1 + 2 + 3 + \ldots + n}{n^2} = \lim_{n\to\infty} \frac{n^2 + n}{2n^2} = \lim_{n\to\infty} \frac{1}{2} + \lim_{n\to\infty} \frac{1}{2n} = \frac{1}{2}.$$

Answer.

a. $\displaystyle \lim_{n\to\infty} \frac{2n^2 + n + \sqrt{n} - 3}{n^2 - 5n + 7} = 2,$ b. $\displaystyle \lim_{n\to\infty} \frac{1 + 2 + 3 + \ldots + n}{n^2} = \frac{1}{2}.$