Solutions of Assignment # 6.

Problem 1. Let $x, y \in \mathbb{R}$ be such that $x \neq y$. Prove that $\exists \varepsilon > 0$ such that $y \notin [x - \varepsilon, x + \varepsilon]$.

Solution. Recall, $y \in [x - \varepsilon, x + \varepsilon]$ means

$$x - \varepsilon \le y \le x + \varepsilon.$$

Assume first $y > x$. Let $\varepsilon = (y - x)/2 > 0$. Then

$$y = x + 2\varepsilon > x + \varepsilon,$$

in particular $y \notin [x - \varepsilon, x + \varepsilon]$.

Now assume $y < x$. Let $\varepsilon = (x - y)/2 > 0$. Then

$$y = x - 2\varepsilon < x - \varepsilon,$$

in particular $y \notin [x - \varepsilon, x + \varepsilon]$.

Note that if both cases we chose $\varepsilon = |(x - y)/2|$. □

Problem 2. Is it true that for every two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ satisfying $\forall n \ \ x_n < y_n$ one has

a. $\sup x_n \le \sup y_n$? **b.** $\sup x_n < \sup y_n$?

Explain your answer (that is, prove if YES; provide a counterexample if NO).

Answer and solution.

a. Yes. Indeed, denote $a = \sup A$ and $b = \sup B$. First, if $a = \infty$ then, by definitions, $\{x_n\}_{n\in\mathbb{N}}$ is unbounded. Therefore, for every M there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} > M$. Hence $y_{n_0} > M$ as well. It shows that $\{y_n\}_{n\in\mathbb{N}}$ is also unbounded. Thus $b = \infty$ as well, and we have $a \le b$.

Now assume $a < \infty$ and also assume that $a > b$. Denote $\varepsilon = (a - b)/2 > 0$. By a theorem, proved in the class, for this ε there exists and element of $\{x_n\}_{n\in\mathbb{N}}$, that is, there exists x_k, such that $a - \varepsilon < x_k$. Using $a > b$ we have $a - \varepsilon = (a + b)/2 > b$. It implies $y_k > x_k > b$, which contradicts to the definition of b (recall $b = \sup B$, in particular, b is an upper bound for $\{y_n\}_{n\in\mathbb{N}}$).

b. No. For every n define $x_n = (n - 1)/n$ and $y_n = 1$. Then for every n we have $x_n < y_n$, but $\sup x_n = \sup y_n = 1$.

Problem 3. A student was trying to recall the definition of a convergent sequence, and came up with the following statements. Your job is to persuade the student that these definitions are wrong, and not even equivalent to the correct definition. In order to do this, for each of these statements you need to find an example of a sequence which either satisfies the statement but fails to converge, or which converges but fails the statement (recall that a sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N \ \forall n \ge N \quad |x_n - \ell| < \varepsilon$).

a. $\forall \varepsilon \ge 0 \ \exists N \in \mathbb{N} \ \forall n \ge N \quad |x_n - \ell| < \varepsilon$;

b. $\forall \varepsilon > 0 \ \exists n \in \mathbb{N} \quad |x_n - \ell| < \varepsilon$;

c. $\exists N \in \mathbb{N} \ \forall \varepsilon > 0 \ \forall n \ge N \quad |x_n - \ell| < \varepsilon$;

d. $\exists \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \ge N \quad |x_n - \ell| < \varepsilon$;

e. $\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n \ge N \quad |x_n - \ell| < \varepsilon$.

Solution. Although for each problem it is enough to provide just ONE sequence, which provides a counterexample (and it will be a complete solution), we provide an analysis of a statement as well.

a. Let us note that NO sequence satisfies the statement. Indeed, the statement fails for $\varepsilon = 0$, since $|x_n - \ell|$ is always nonnegative. Formally, for any sequence we have $\exists \varepsilon \geq 0$ (namely $\varepsilon = 0$) such that $\forall N \in \mathbb{N} \;\; \exists n \geq N$ (namely $n = N$) such that $|x_n - \ell| \geq 0$. Thus it is enough to provide a sequence, which is convergent. We can use any example from the class or to take a constant sequence, say $x_n = 0$ for every n. Then 0 is clearly limit of the sequence ($\forall \varepsilon > 0 \; \exists N$, namely $N = 1$, such that $\forall n \geq N \quad |x_n - 0| = 0 < \varepsilon$).

b. Consider $x_n = (-1)^n$ and $\ell = 1$. Then $\forall \varepsilon > 0 \; \exists n \in \mathbb{N}$ (namely $n = 2$) such that $|x_n - \ell| = 0 < \varepsilon$. But $x_n = (-1)^n$ is divergent as was proved in the class. (Note also that with such definition as in **"b"** ANY element of a given sequence will be its limit. Indeed, fix $k \in \mathbb{N}$ and let $\ell = x_k$. Then for every ε there exist n, namely $n = k$, such that $|x_n - \ell| = 0 < \varepsilon$).

c. Let us first note that if $y \geq 0$ satisfies $\forall \varepsilon > 0 \quad |y| < \varepsilon$ then $y = 0$. Indeed, taking $\varepsilon = 1/k$ we obtain that $\forall k \quad |y| < 1/k$. By a theorem in the class it implies that $|y| = 0$. By another theorem we obtain $y = 0$. Therefore

$$\forall \varepsilon > 0 \; \forall n \geq N \quad |x_n - \ell| < \varepsilon$$

implies

$$\forall n \geq N \quad x_n = \ell.$$

So, it is enough to take any convergent sequence which does NOT have a constant tail. For example take $x_n = n/(n+1)$ for every n. In the class we showed that 1 is the limit. Clearly, $|x_n - 1| = 1/n > 0$ for every n, which implies that there is no $N \in \mathbb{N}$ satisfying the statement.

d. Here note that any FIXED positive ε is separated from 0, so the statement does not give that x_n "approaches" ℓ. Moreover, one can take "big" ε.

Take again $x_n = (-1)^n$ for every n. It is divergent sequence, but $\exists \varepsilon > 0$, namely $\varepsilon = 2$, such that $\ell = 0$ satisfies the statement (since $\forall n \in \mathbb{N} \quad |x_n - \ell| < 2$).

Note here three facts. Firstly, any bounded sequence would satisfy this statement (**exer.**) Secondly, any bounded sequence will have infinitely many limits with such a definition (**exer.**). Finally, even if we ask "there exists a small ε", say $0 < \varepsilon < 10^{-3}$, similar examples will work (**exer.**).

e. Note that the condition $\exists n \geq N$... does not say anything about the whole tail, only about a SINGLE representative of a tail.

Consider again $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = (-1)^n$ and $\ell = 1$. Then, as we mentioned before, $\{x_n\}_{n=1}^{\infty}$ is divergent. But $\forall \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N$ (namely any even n bigger than N works, say $n = 2N$) such that $|x_n - \ell| = 0 < \varepsilon$

Note here that with such a definition a sequence can have many limits (in particular in our example we would have two limits, namely -1 and 1). □

Problem 4. Find limits (and then prove using the definition) of the following sequences.

$$\textbf{a.} \quad \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}, \qquad \textbf{b.} \quad \left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}, \qquad \textbf{c.} \quad \left\{\frac{(n+2)^2}{n^2}\right\}_{n=1}^{\infty}.$$

Solution.

a. We show that the sequence converges to 0. Let $\varepsilon > 0$. Choose $N > 1/\varepsilon$. Then for every $n \geq N$ we have

$$|x_n - 0| = \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus $\forall \varepsilon > 0\ \exists N$ (namely any $N > 1/\varepsilon$ works) such that $\forall n \geq N \quad |x_n - 0| < \varepsilon$.

b. We show that the sequence converges to 0. Let $\varepsilon > 0$. Choose $N > 1/\varepsilon$. Then for every $n \geq N$ we have

$$|x_n - 0| = \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus $\forall \varepsilon > 0\ \exists N$ (namely any $N > 1/\varepsilon$ works) such that $\forall n \geq N \quad |x_n - 0| < \varepsilon$.

c. We show that the sequence converges to 1. Let $\varepsilon > 0$. Choose $N > 8/\varepsilon$. Then for every $n \geq N$ we have

$$\left| \frac{(n+2)^2}{n^2} - 1 \right| = \frac{n^2 + 4n + 4 - n^2}{n^2} = \frac{4}{n} + \frac{4}{n^2} \leq \frac{8}{n} \leq \frac{8}{N} < \varepsilon.$$

Thus $\forall \varepsilon > 0\ \exists N$ (namely any $N > 8/\varepsilon$ works) such that $\forall n \geq N \quad |x_n - 1| < \varepsilon$.

Answer.

a. $\lim\limits_{n \to \infty} \dfrac{1}{n^2} = 0$, **b.** $\lim\limits_{n \to \infty} \dfrac{(-1)^n}{n} = 0$, **c.** $\lim\limits_{n \to \infty} \dfrac{(n+2)^2}{n^2} = 1$.