Solutions of Assignment # 5.

Problem 1. Let $r > 0$. Prove that

$$|x| < r \ \Leftrightarrow\ -r < x < r.$$

Solution.
Part 1. $|x| < r \implies -r < x < r$. Assume $|x| < r$.
Case a. $x \geq 0$. In this case $|x| = x$. Hence we have $0 \leq x = |x| < r$. It implies $-r < x < r$.
Case b. $x < 0$. In this case $|x| = -x$. Hence we have $-x < r$, which implies $x > -r$. Thus, in this case, $-r < x < 0$, therefore $-r < x < r$.

Part 2. $-r < x < r \implies |x| < r$. Assume $-r < x < r$. If $x \geq 0$ then $|x| = x < r$. If $x < 0$ then $|x| = -x$. Since, by our assumption, $x > -r$, we obtain $-x < r$. It shows that that for $x < 0$ we also have $|x| < r$. $\square$

Problem 2. Solve and write the answer in the interval notation.

$$\textbf{a.}\quad \left|\frac{3}{2x-1}\right| < 1, \qquad\qquad \textbf{b.}\quad \left|\frac{1}{1-3x}\right| > 4.$$

Solution.
a. First note that for $x = 1/2$ the fraction $\frac{3}{2x-1}$ is not defined (and thus $x = 1/2$ is not a solution). Now we consider the case $x \neq 1/2$ (and all equivalences will be under this condition).

$$\left|\frac{3}{2x-1}\right| = \frac{3}{|2x-1|} < 1 \ \Leftrightarrow\ 3 < |2x-1|\,.$$

Case 1. $2x - 1 \geq 0$. In this case $|2x-1| = 2x-1$, so we have

$$3 < 2x - 1 \ \Leftrightarrow\ 2 < x$$

(note that any $x > 2$ satisfies the conditions $2x - 1 \geq 0$ and $x \neq 1/2$).
Case 2. $2x - 1 < 0$. In this case $|2x-1| = -(2x-1)$, so we have

$$3 < -2x + 1 \ \Leftrightarrow\ 2 < -2x \ \Leftrightarrow\ x < -1.$$

(note that any $x < -1$ satisfies the condition $2x - 1 < 0$ and $x \neq 1/2$).

Combining two cases we obtain $x \in (-\infty, -1) \cup (2, \infty)$. (Note again that both intervals do not contain $1/2$, so condition $x \neq 1/2$ is satisfied).
b. First note that for $x = 1/3$ the fraction $\frac{1}{1-3x}$ is not defined (and thus $x = 1/3$ is not a solution). Now we consider the case $x \neq 1/3$ (and all equivalences will be under this condition).

$$\left|\frac{1}{1-3x}\right| = \frac{1}{|3x-1|} > 4 \ \Leftrightarrow\ 1/4\ > |3x-1| \ \Leftrightarrow\ -1/4 < 3x - 1 < 1/4 \ \Leftrightarrow\ 3/4 < 3x < 5/4,$$

which means $x \in (1/4, 5/12)$ Combining this with our condition $x \neq 1/3$, we obtain that $x \in (1/4, 1/3) \cup (1/3, 5/12)$. $\square$

Answer. **a.** $(-\infty, -1) \cup (2, \infty)$, **b.** $(1/4, 1/3) \cup (1/3, 5/12)$.

Problem 3. Let $S \neq \emptyset$ be a bonded (below and above) subset of $\mathbb{R}$. Denote $a = \inf S$, $b = \sup S$. Is it true that $\forall n \in \mathbb{N}\ \ \exists x \in S$ such that

a. $a > x - \frac{1}{n}$; **b.** $a > x + \frac{1}{n}$; **c.** $b < x - \frac{1}{n}$; **d.** $b < x + \frac{1}{n}$.

Answer and Solution. We use that all numbers n in $\mathbb{N}$ (and therefore $1/n$) are positive.

a. Yes. We had a theorem in class that $\forall \varepsilon > 0 \ \ \exists x \in S$ such that $a > x - \varepsilon$. Applying this theorem with $\varepsilon = 1/n$ we observe the desired result.

b. No. Indeed, by the definition, a is a lower bound, that is, for every $x \in S$ one has $a \leq x$. In particular, $a < x + 1/n$ for all $n \in \mathbb{N}$ and all $x \in S$.

c. No. Indeed, by the definition, b is an upper bound, that is, for every $x \in S$ one has $b \geq x$. In particular, $b > x - 1/n$ for all $n \in \mathbb{N}$ and all $x \in S$.

d. Yes. We had a theorem in class that $\forall \varepsilon > 0 \ \ \exists x \in S$ such that $b < x + \varepsilon$. Applying this theorem with $\varepsilon = 1/n$ we observe the desired result. $\square$

Problem 4. For a given set S write $\sup S$, $\inf S$, $\max S$, $\min S$ (if exist).

a. $S = \{-1, 2, 5\}$; **b.** $S = [1, 2]$; **c.** $S = (2, 3]$;
d. $S = [0, 4)$; **e.** $S = (-\infty, 3]$; **f.** $S = (1, \infty)$;
g. $S = \mathbb{N}$; **h.** $S = \mathbb{Z}$; **i.** $S = \emptyset$.

Answers.

a. $\sup S = \max S = 5$, $\inf S = \min S = -1$;
b. $\sup S = \max S = 2$, $\inf S = \min S = 1$;
c. $\sup S = \max S = 3$, $\inf S = 2$, $\min S$ does not exist;
d. $\sup S = 4$, $\inf S = \min S = 0$, ; $\max S$ does not exist;
e. $\sup S = \max S = 3$, $\inf S = -\infty$, ; $\min S$ does not exist;
f. $\sup S = \infty$, $\inf S = 1$ $\max S$ and $\min S$ do not exist;
g. $\sup S = \infty$, $\inf S = \min S = 1$, $\max S$ does not exist;
h. $\sup S = \infty$, $\inf S = -\infty$, $\max S$ and $\min S$ do not exist;
i. $\sup S = -\infty$, $\inf S = \infty$, $\max S$ and $\min S$ do not exist.