Solutions of Assignment # 4.

Problem 1. Solve and write the answer in the interval notation.
a. $|3+2x|<5$; **b.** $|3-2x|\le 5$.

Solution.
a.
$$|3+2x|<5 \;\Leftrightarrow\; -5<3+2x<5 \;\Leftrightarrow\; -8<2x<2 \;\Leftrightarrow\; -4<x<1.$$

b.
$$|3-2x|\le 5 \;\Leftrightarrow\; -5\le 3-2x\le 5 \;\Leftrightarrow\; -8\le -2x\le 2 \;\Leftrightarrow\; 4\ge x\ge -1.$$

□

Answer. **a.** $(-4,1)$, **b.** $[-1,4]$.

Remark. Another (longer) way to solve this problem is to use the definition of absolute value, that is, considering cases. Please note, writing the result of a current case, never forget to check the condition of the case. For example, the solution of **a.** can be written as
Case 1. $3+2x\ge 0$. First note that this case is
$$3+2x\ge 0 \;\Leftrightarrow\; 2x\ge -3 \;\Leftrightarrow\; x\ge -1.5.$$
In this case, $|3+2x|=3+2x$, so we have to solve
$$3+2x<5 \;\Leftrightarrow\; 2x<2 \;\Leftrightarrow\; x<1.$$
Combining it with our condition on x (the condition of Case 1), we have $x\ge -1.5$ and $x<1$, or, in the interval conditions, $x\in[-1.5,1)$.
Case 2. $3+2x<0$. In this case is
$$3+2x<0 \;\Leftrightarrow\; 2x<-3 \;\Leftrightarrow\; x<-1.5.$$
In this case, $|3+2x|=-(3+2x)=-3-2x$, so we have to solve
$$-3-2x<5 \;\Leftrightarrow\; -2x<8 \;\Leftrightarrow\; x>-4.$$
Combining it with our condition on x (the condition of Case 2), we have $x<-1.5$ and $x>-4$, or, in the interval conditions, $x\in(-4,-1.5)$.

From Case 1 and Case 2, we obtain the solution, $x\in(-4,-1.5)\cup[-1.5,1)=(-4,1)$.

Problem 2. Is it true that **a.** $x\le 2$ implies $|x|\le 2$?
b. $|x|\le 4$ implies $x\le 4$? **c.** $|x|\le 9$ implies $x\ge -5$?
d. $x>1$ implies $|x|>1$? **e.** $|x|>3$ implies $x>3$?
(Please note that ($P(x)$ implies $Q(x)$) means ($\forall x\in\mathbb{R}\ (P(x) \implies Q(x))$).)

Answer and Solution.
a. No. Indeed, take $x=-5$. Then $x\le 2$, but $|x|=5>2$.
b. Yes. Indeed if $x>4$ then $x>0$, so $|x|=x>4$. (Note, we used that the statement "P implies Q" is equivalent to the statement "(not Q) implies (not P)". One can also argue by contradiction (which is very similar): Assume not. Assume that that there exists x such that $|x|\le 4$ and $x>4$. Since $x>4>0$ we have $|x|=x>4$. It contradicts to $|x|\le 4$.)
c. No. Take $x=-9$. Then $|x|=9\le 9$, but $x<-5$. (However, note that $|x|\le 9$ implies $x\ge -15$. Indeed, $|x|\le 9 \;\Leftrightarrow\; -9\le x\le 9$. In particular, $x\ge -9>-15$.)

d. Yes. If $x > 1$ then $x > 0$, so we have $|x| = x > 1$.
e. Not. Take $x = -4$. Then $|x| = 4 > 3$ but $x \leq 3$. $\square$

Problem 3. Is it true that for every real numbers x, y, z one has $|x - z| \leq |x - y| + |y - z|$?

Answer and Solution. Yes. Indeed, denote $u = x - y$, $v = y - z$. Then we have to check that $|u + v| \leq |u| + |v|$, which is true by the triangle inequality, proved in the class. $\square$

Problem 4. For a set S given below describe the set of all upper bounds of S and the set of all lower bounds of S. If possible, write the answer in the interval notation.

a. $S = \{-1, 2, 5\}$; **b.** $S = [1, 2]$; **c.** $S = (2, 3]$;
d. $S = [0, 4)$; **e.** $S = (-\infty, 3]$; **f.** $S = (1, \infty)$;
g. $S = \mathbb{R}$; **h.** $S = \emptyset$.

Answers. We denote

$$A = \{x \,|\, x \text{ is a lower bound of } S\} \qquad \text{and} \qquad B = \{x \,|\, x \text{ is an upper bound of } S\}.$$

a. $A = (-\infty, -1]$, $B = [5, \infty)$;
b. $A = (-\infty, 1]$, $B = [2, \infty)$;
c. $A = (-\infty, 2]$, $B = [3, \infty)$;
d. $A = (-\infty, 0]$, $B = [4, \infty)$;
e. $A = \emptyset$, $B = [3, \infty)$;
f. $A = (-\infty, 1]$, $B = \emptyset$;
g. $A = \emptyset$, $B = \emptyset$;
h. $A = \mathbb{R}$, $B = \mathbb{R}$.