Solutions of Assignment # 3.

Problem 1. Using the induction principle prove that for every $x > 0$ and every $n \in \mathbb{N}$ one has $(1+x)^n \geq 1 + x^n$.

Solution.
1 (base). If $n = 1$ we have $(1+x)^1 \geq 1 + x^1$, which is clearly true (actually we have an equality).
2 (step). Assume the formula holds for n, that is $(1+x)^n \geq 1 + x^n$. Then

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+x^n)(1+x) = 1 + x^n + x + x^{n+1} \geq 1 + x^{n+1}.$$

In the first inequality above we used the induction assumption and $x > 0$ (so $1 + x > 0$), in the second one we used that $x^n + x \geq 0$ for every $n \geq 1$ and $x > 0$. This proves the induction step and, thus, concludes the proof of the general formula. □

Problems 2. Let a, b be non-zero real numbers. Using only definitions and axioms prove
a. $(a^{-1})^{-1} = a$; **b.** $\frac{1}{a} = a^{-1}$; **c.** $(ab)^{-1} = a^{-1}b^{-1}$;
d. $a > 0$ if and only if $a^{-1} > 0$; **e.** $a > 1$ if and only if $0 < a^{-1} < 1$.

Solution. Note that according to the definitions **a.** to check that $x = -u$ one has to check that $u + x = 0$; **b.** to check that $y = u^{-1}$ one has to check that $u \cdot y = 1$.
a. Denote $u = a^{-1}$. We want to show that $a = u^{-1}$. Applying A5 and the definition of a^{-1} we obtain

$$u \cdot a = a^{-1} \cdot a = a \cdot a^{-1} = 1.$$

By the definition, it shows that $a = u^{-1}$ as needed.
b. Recall that by definition $\frac{x}{y} = x \cdot y^{-1}$. Thus, using this definition and the definition of 1, we observe

$$\frac{1}{a} = 1 \cdot a^{-1} = a^{-1}.$$

c. Denote $w = a \cdot b$. Applying A5 and A6 a few times and the definitions of 1, a^{-1}, b^{-1}, we obtain

$$w \cdot (a^{-1} \cdot b^{-1}) = w \cdot (b^{-1} \cdot a^{-1}) = (w \cdot b^{-1}) \cdot a^{-1} = ((a \cdot b) \cdot b^{-1}) \cdot a^{-1}$$

$$= (a \cdot (b \cdot b^{-1})) \cdot a^{-1} = (a \cdot 1) \cdot a^{-1} = a \cdot a^{-1} = 1$$

By the definition, it shows that $a^{-1} \cdot b^{-1} = w^{-1}$ as needed.
d.
1. First we show that $a > 0$ implies $a^{-1} > 0$. We argue by contradiction. Assume that $a > 0$ but $a^{-1} < 0$. Then $a^{-1} \leq 0$ and $0 \leq a$, so, by axiom O6 we observe that $a^{-1} \cdot a \leq 0 \cdot a$. By A5, the definition of a^{-1}, and since $a \cdot 0 = 0$, we obtain $1 \leq 0$, which contradicts to $1 > 0$. It proves the first part.
2. Now we show that $a^{-1} > 0$ implies $a > 0$ (the proof is essentially the same; please note that instead of the prove given below, we could apply the first part of the statement to the real number $b = a^{-1}$ and then to say that $a = b^{-1}$ which was proved in part **a.**). We argue by contradiction. Assume that $a^{-1} > 0$ but $a < 0$. Then $a \leq 0$ and $0 \leq a^{-1}$, so, by axiom O6 we observe that $a \cdot a^{-1} \leq 0 \cdot a^{-1}$. By A5 the definition of a^{-1}, and since $a^{-1} \cdot 0 = 0$, we obtain $1 \leq 0$, which contradicts to $1 > 0$. It proves the result.
e.
1. First we show that $a > 1$ implies $0 < a^{-1} < 1$. Assume that $a > 1$. Since $1 > 0$, we observe by O3 that $a > 0$ (in fact O3 gives $a \geq 0$, but, if $a = 0$ then $0 = a > 1$). By the part **d.** we get that $a^{-1} > 0$. Therefore it remains to prove that $a^{-1} < 1$. Assume by contradiction that $a^{-1} \geq 1$.

Multiply this inequality by a non-negative number a. By O6 we obtain $a^{-1} \cdot a \geq 1 \cdot a$. By A5 and definitions it implies $1 \geq a$, which contradicts to our assumption. Therefore $a^{-1} < 1$.
2. Now we show that $0 < a^{-1} < 1$ implies $a > 1$. Assume by contradiction that $0 < a^{-1} < 1$ but $a \leq 1$. By O6 multiply the latter inequality by a non-negative number a^{-1}. We obtain $a \cdot a^{-1} \leq 1 \cdot a^{-1}$. By definitions, $1 \leq a^{-1}$, which contradicts our assumption. It proves the result. □

Recall order axioms and some definitions

(for field axioms see the solutions of assignment 2.)

O1. $\forall a \in \mathbb{R}$ $a \leq a$;
O2. $\forall a, b \in \mathbb{R}$ if $a \leq b$ and $b \leq a$ then $a = b$;
O3. $\forall a, b, c \in \mathbb{R}$ if $a \leq b$ and $b \leq c$ then $a \leq c$;
O4. $\forall a, b \in \mathbb{R}$ $a \leq b$ or $b \leq a$;
O5. $\forall a, b, c \in \mathbb{R}$ if $a \leq b$ then $a + c \leq b + c$;
O6. $\forall a, b, c \in \mathbb{R}$ if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

Definitions. We denote $a < b$ if $a \leq b$ and $a \neq b$. We write $a \geq b$ if $b \leq a$. We write $a > b$ if $b < a$.