Solutions of Assignment # 2.

Problem 1. Using the induction principle prove that
a. for every integer $n \geq 1$ one has

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2;$$

b. $8^n - 3^n$ is divisible by 5 for every integer $n \geq 1$;
c. for every integer $n \geq 2$ one has

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) < n^{2n}.$$

Solution.
a. **1 (base).** If $n = 1$ we have $1^3 = (1 \cdot 2/2)^2$, which is true $(1 = 1)$.
2 (step). Assume the formula holds for n. Denote $m = n + 1$. Then, using induction assumption we obtain,

$$1^3 + 2^3 + 3^3 + \ldots + m^3 = (1^3 + 2^3 + 3^3 + \ldots + n^3) + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{(n+1)^2}{4}\left(n^2 + 4(n+1)\right) = \frac{(n+1)^2}{4}(n+2)^2 = \left(\frac{m(m+1)}{2}\right)^2.$$

This proves the induction step and, thus, concludes the proof of the formula in the general case.
b. **1 (base).** If $n = 1$ we have $8^1 - 3^1 = 5$, which is divisible by 5.
2 (step). Assume that $8^n - 3^n$ is divisible by 5, that is $8^n - 3^n = 5k$ for some (positive) integer k. Then $8^n = 5k + 3^n$. Thus, for $n + 1$, we have

$$8^{n+1} - 3^{n+1} = 8 \cdot 8^n - 3 \cdot 3^n = 8(5k + 3^n) - 3 \cdot 3^n = 8 \cdot 5k + 5 \cdot 3^n = 5(8k + 3^n).$$

Since $8k + 3^n$ is integer, we obtain that $8^{n+1} - 3^{n+1}$ is divisible by 5, i.e. we proved the induction step. Thus we proved the statement.
c. **1 (base).** If $n = 2$ then $2n + 1 = 5$ and we have

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) = 1 \cdot 3 \cdot 5 = 15,$$

while $n^{2n} = n^{2n} = 2^4 = 16$. Therefore the inequality is true for $n = 2$.
2 (step). Assume that $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) < n^{2n}$ and denote $m = (n+1)$. Then, by assumption,

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m+1) = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1) \cdot (2n+3) < n^{2n}(2n+3) \leq (n+1)^{2n}(2n+3)$$

and $m^{2m} = (n+1)^{2n+2} = (n+1)^{2n}(n^2 + 2n + 1)$. Thus, in order to prove,

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m+1) < m^{2m},$$

it is enough to prove

$$2n + 3 \leq n^2 + 2n + 1.$$

The latter is equivalent to

$$2 \leq n^2,$$

which is true for every $n \geq 2$. It completes the induction step and, thus, the inequality is proved. $\square$

Problems 2. Using only definitions and axioms A1 – A4 prove that
a. $-0 = 0$; **b.** $0 - 1 = -1$

Solution. Recall that, by the definition of the inverse element, in order to prove that $b = -x$ it is enough to show that $x + b = 0$.
a. In this case, $x = b = 0$. By axiom A3 we have $0 + 0 = 0$, hence $0 = -0$.
b. In this case $x = 1$ and $b = 0 - 1$. We have

$$1 + (0 - 1) = 1 + (0 + (-1)) = 1 + ((-1) + 0) = (1 + (-1)) + 0 = 0 + 0 = 0,$$

where in the first equality we used the definition of the operation "$-$", in the second one we used axiom A1, in the third one we used axiom A2, in the fourth one we used the definition of the inverse element, and in the fifth equality axiom A3 was used. It proves the desired result. $\square$

Problem 3. Using only definitions and axioms A1 – A9 prove that
a. for every $a, b \in \mathbb{R}$ one has $(-a)b = -(ab)$;
b. for every $a \in \mathbb{R}$ one has $(-1)a = -a$.
(In this problem you may also use the fact saying that $x0 = 0$ for every $x \in \mathbb{R}$).

Solution. Recall again that in order to prove that $y = -x$ it is enough to show that $x + y = 0$.
a. Here $x = ab$, $y = (-a)b$. We have

$$ab + (-a)b = ba + b(-a) = b(a + (-a)) = b \cdot 0 = 0,$$

where in the first equality we used axiom A5, in the second one we used axiom A9, in the third one we used the definition of the inverse element, and in the fourth equality we used fact, proved in the class.
b. Way 1. Here $x = a$ and $y = (-1)a$. We have

$$a + (-1)a = a \cdot 1 + a(-1) = a(1 + (-1)) = a \cdot 0 = 0,$$

where in the first equality we used axioms A7 and A5, in the second one we used axiom A9, in the third one we used the definition of the inverse element, and in the fourth equality we used fact, proved in the class.
b. Way 2. Note that in the problem **3a** we proved $(-a)b = -(ab)$ for every $a, b \in \mathbb{R}$. Applying that with $a = 1$ we observe that $(-1)b = -(1 \cdot b)$ for every $b \in \mathbb{R}$. Since by axioms A7 and A5 $1 \cdot b = b \cdot 1 = b$, we obtain $(-1)b = -b$ for every $b \in \mathbb{R}$, which proves the result. $\square$

Recall axioms (and Fact 4).

A1. $\forall a, b \in \mathbb{R} \quad a + b = b + a$;
A2. $\forall a, b, c \in \mathbb{R} \quad (a + b) + c = a + (b + c)$;
A3. $\exists \theta \in \mathbb{R} \quad a + \theta = a$;
A4. $\forall a \in \mathbb{R} \quad \exists x \in \mathbb{R} \quad a + x = \theta$;
A5. $\forall a, b \in \mathbb{R} \quad a \cdot b = b \cdot a$;
A6. $\forall a, b, c \in \mathbb{R} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$;
A7. $\exists z \in \mathbb{R} \quad a \cdot z = a$;
A8. $\forall a \in \mathbb{R} \setminus \{\theta\} \ \exists y \in \mathbb{R} \quad a \cdot y = z$;
A9. $\forall a, b, c \in \mathbb{R} \quad a \cdot (b + c) = a \cdot b + a \cdot c$.

Fact 4. $\forall a \in \mathbb{R} \quad a \cdot 0 = 0$.

Definitions. The unique element θ given by $A3$ (and used in A4) is denoted by 0. Given a, the unique element x given by $A4$ is denoted by $-a$. The unique element z given by $A7$ (and used in A8) is denoted by 1. Given $a \neq 0$, the unique element y given by $A8$ is denoted by a^{-1}. **Note** that according to this definition **a.** to check that $x = -a$ one has to check that $a + x = 0$; **b.** to check that $y = a^{-1}$ one has to check that $a \cdot y = 1$.