Solutions of Assignment # 11.

Problem 1. Find

$$\lim_{x\to a^-} f(x), \qquad \lim_{x\to a^+} f(x), \qquad \lim_{x\to a} f(x)$$

if exist. As usual, justify your answer.

a. $a = 0, \quad f(x) = \begin{cases} x^3, & \text{if } x \ge 1, \\ (x-1)^{-2}, & \text{if } 0 < x < 1, \\ |x-1|, & \text{if } x \le 0; \end{cases}$

b. $a = 1, \quad f(x) = \begin{cases} (x-1)^{-2}, & \text{if } x > 1, \\ |x-1|, & \text{if } x < 1; \end{cases}$

c. $a = 3, \quad f(x) = \dfrac{x+3}{x-3}.$

Solution.
a. We show that

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = 1.$$

First consider $x_n \to 0^+$ as $n \to \infty$, that is $x_n > 0$ (for every n) and $x_n \to 0$ as $n \to \infty$. Then, by the definition of the limit (applied with $\varepsilon = 1$) there exists N such that for every $n \ge N$ one has $0 < x_n < 1$. Then $f(x_n) = (x_n - 1)^{-2}$. From corresponding results for limits of sequences we obtain that $\{f(x_n)\}_{n=N}^{\infty}$ is convergent to 1. It implies that $f(x_n) \to 1$ as $n \to \infty$. Therefore,

$$\lim_{x\to 0^+} f(x) = 1.$$

Now consider $x_n \to 0^-$ as $n \to \infty$, that is $x_n < 0$ (for every n) and $x_n \to 0$ as $n \to \infty$. Then $f(x_n) = |x_n - 1|$. From corresponding results for limits of sequences we obtain that $f(x_n) \to 1$ as $n \to \infty$. Therefore,

$$\lim_{x\to 0^-} f(x) = 1.$$

By a theorem in the class if one-sided limits exist and equal to each other, the limit is also exist and equal to the same number. Thus we proved also

$$\lim_{x\to 0} f(x) = 1.$$

b. We show that

$$\lim_{x\to a^-} f(x) = 0, \qquad \lim_{x\to a^+} f(x) = \infty, \qquad \lim_{x\to a} f(x) \quad \text{does not exist.}$$

First consider $x_n \to 1^+$ as $n \to \infty$, that is $x_n > 1$ (for every n) and $x_n \to 1$ as $n \to \infty$. Then $f(x_n) = (x_n - 1)^{-2}$. From corresponding results for limits of sequences we obtain that $|(x_n - 1)^2| = (x_n - 1)^2 \to 0$ as $n \to \infty$ and, by another theorem, it implies that $f(x_n) \to \infty$ as $n \to \infty$. Therefore,

$$\lim_{x\to 0^+} f(x) = \infty.$$

Now consider $x_n \to 1^-$ as $n \to \infty$, that is $x_n < 1$ (for every n) and $x_n \to 1$ as $n \to \infty$. Then $f(x_n) = |x_n - 1|$. From corresponding results for limits of sequences we obtain that $f(x_n) \to 0$ as $n \to \infty$. Therefore,

$$\lim_{x\to 1^-} f(x) = 0.$$

Since one-sided limits are not equal, the limit at 1 does not exists.
c. We show that

$$\lim_{x\to a^-} f(x) = -\infty, \qquad \lim_{x\to a^+} f(x) = \infty, \qquad \lim_{x\to a} f(x) \quad \text{does not exist.}$$

First note, that if $x_n \to 3$ as $n \to \infty$ then $|x_n - 3| \to 0$ as $n \to \infty$. If, in addition we have $x_n \neq 3$ (for every n) then by a theorem in the class we obtain

$$\frac{1}{|x_n - 3|} \to \infty \quad \text{as} \quad n \to \infty.$$

By another theorem in the class, since $x_n + 3 \to 6$ as $n \to \infty$, we obtain that

$$\frac{x_n + 3}{|x_n - 3|} \to \infty \quad \text{as} \quad n \to \infty.$$

It implies that
Case 1. If $x_n \to 3^+$ as $n \to \infty$, that is $x_n > 3$ (for every n) and $x_n \to 3$ as $n \to \infty$ then

$$f(x) = \left|\frac{x_n + 3}{x_n - 3}\right| \to \infty \quad \text{as} \quad n \to \infty;$$

Case 2. If $x_n \to 3^-$ as $n \to \infty$, that is $x_n < 3$ (for every n) and $x_n \to 3$ as $n \to \infty$ then

$$f(x) = -\left|\frac{x_n + 3}{x_n - 3}\right| \to -\infty \quad \text{as} \quad n \to \infty;$$

Since one-sided limits are not equal, the limit at 3 does not exists. □

Answer.

a. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = 1.$

b. $\lim_{x\to a^-} f(x) = 0, \qquad \lim_{x\to a^+} f(x) = \infty, \qquad \lim_{x\to a} f(x)$ does not exist.

c. $\lim_{x\to a^-} f(x) = -\infty, \qquad \lim_{x\to a^+} f(x) = \infty, \qquad \lim_{x\to a} f(x)$ does not exist.

Problem 2. Find

$$\lim_{x\to -\infty} f(x), \qquad \lim_{x\to \infty} f(x)$$

if exist. As usual, justify your answer.

$$\textbf{a.} \quad f(x) = \frac{x^2 + 1}{x^3 - 1}, \qquad\qquad \textbf{b.} \quad f(x) = \frac{x^4 - 1}{1 - x^2}.$$

Solution.
a. We show that

$$\lim_{x\to -\infty} f(x) = \lim_{x\to \infty} f(x) = 0.$$

First note that if $x_n \to \pm\infty$ as $n \to \infty$ then $|x_n| \to \infty$ as $n \to \infty$. Note also that if $|x| > 2$ then

$$x^2 + 1 \le 2x^2 \quad \text{and} \quad |x^3 - 1| \ge |x^3| - 1 \ge |x^3|/2.$$

Thus, for $|x| > 2$ we have

$$\left|\frac{x^2+1}{x^3-1}\right| \le \frac{2x^2}{|x^3|/2} = \frac{4}{|x|}.$$

Now, assume $x_n \to \pm\infty$ as $n \to \infty$, so $|x_n| \to \infty$ as $n \to \infty$. By the definition (applied with $M = 2$) there exists N such that for every $n \ge N$ one has $|x_n| \ge 2$, which implies that

$$0 \le |f(x_n)| \le \frac{4}{|x_n|} \to 0 \quad \text{as} \quad n \to \infty.$$

By the Squeeze Theorem it implies that $|f(x_n)| \to 0$ as $n \to \infty$, which proves that $f(x_n) \to 0$ as $n \to \infty$. It implies the result.

b. We show that

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = -\infty.$$

First note that if $x_n \to \pm\infty$ as $n \to \infty$ then $x_n^4 \to \infty$ and $x_n^2 \to \infty$ as $n \to \infty$. Note also that if $|x| > 2$ then $x^4 - 1 \ge x^4/2$ and $x^2 - 1 > x^2$, so

$$\frac{x^4-1}{x^2-1} \ge \frac{x^4}{2x^2} = \frac{x^2}{2}.$$

Now, assume $x_n \to \pm\infty$ as $n \to \infty$, so $x_n^4 \to \infty$ and $x_n^2 \to \infty$ as $n \to \infty$. By the definition (applied with $M = 2$) there exists N such that for every $n \ge N$ one has $|x_n| \ge 2$, which implies that

$$-f(x_n) = \left|\frac{x_n^4-1}{1-x_n^2}\right| = \frac{x_n^4-1}{x_n^2-1} \ge \frac{x_n^2}{2} \to \infty \quad \text{as} \quad n \to \infty.$$

It implies that $-f(x_n) \to -\infty$ as $n \to \infty$. $\square$

Answer.

a. $\displaystyle \lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0.$

b. $\displaystyle \lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = -\infty.$

Problem 3. Is the following statement true or false? As usual, justify your answer.

a. If a function f is discontinuous at point a then so is f^2.

b. If a function f is discontinuous at point a and a function g is continuous at point a then $f + g$ is is discontinuous at point a.

Solution.

a. No. Consider f defined by $f(x) = 1$ for $x \ge 0$, $f(x) = -1$ for $x < 0$. Clearly f is discontinuous at 0 (we proved in class that f has no limit at 0), but for every x one has $f^2(x) = 1$, so f^2 is continuous at 0.

b. Yes. Indeed, assume not, that is assume that there are functions f, g, and $h = f + g$ such that g, h are continuous at point a and f is discontinuous at point a. By the theorem in the class we observe that $f = h - g$ is continuous at point a. Contradiction. $\square$

Answer. **a.** No. **b.** Yes.