Solutions of Assignment # 10.

Problem 1. Let $k, \ell \in \mathbb{N}$ be such that $k > \ell > 2$. Find the following limits.

$$\textbf{a.} \quad \lim_{n\to\infty} \frac{n^k+1}{10n^\ell+5n^2+n}, \qquad \textbf{b.} \quad \lim_{n\to\infty} \frac{n^{k+\ell}+n^k+n^\ell}{2^n}, \qquad \textbf{c.} \quad \lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}}.$$

Solution.
a. By theorem proved in class we have

$$\lim_{n\to\infty} \frac{n^k+1}{10n^\ell+5n^2+n} = \lim_{n\to\infty} \frac{n^k}{10n^\ell} = \infty.$$

b. By theorems proved in class we have

$$\lim_{n\to\infty} \frac{n^{k+\ell}+n^k+n^\ell}{2^n} = \lim_{n\to\infty} \frac{n^{k+\ell}}{2^n} + \lim_{n\to\infty} \frac{n^k}{2^n} + \lim_{n\to\infty} \frac{n^\ell}{2^n} = 0+0+0 = 0.$$

b. In class we proved

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \qquad (*).$$

It implies that

$$\lim_{n\to\infty} (\sqrt{n})^{\frac{1}{\sqrt{n}}} = 1 \qquad (**)$$

(this fact can be proved using the same lines as for $(*)$, we provide the proof below. Note here that the sequence in $(**)$ is not a subsequence of the sequence in $(*)$).

Therefore, by one of the properties of limit (which was a homework problem) we obtain

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} (\sqrt{n})^{\frac{2}{\sqrt{n}}} = 1^2 = 1.$$

Proof of $(**)$. Denote the sequence in $(**)$ by $\{x_n\}_{n=1}^\infty$. First we show that this sequence is decreasing starting from $n = 27$. Note

$$\frac{\sqrt{n+1}}{\sqrt{n}} - 1 = \sqrt{1+\frac{1}{n}} - 1 = \frac{1+1/n-1}{\sqrt{1+1/n}+1} = \frac{1}{n\ \sqrt{1+1/n}+1} \geq \frac{1}{3n}. \qquad (***)$$

Thus

$$(\sqrt{n+1})^{\frac{1}{\sqrt{n+1}}} \leq (\sqrt{n})^{\frac{1}{\sqrt{n}}} \ \Leftrightarrow\ n+1 \leq n^{\frac{\sqrt{n+1}}{\sqrt{n}}} = n\ n^{\frac{\sqrt{n+1}}{\sqrt{n}}-1} \ \Leftrightarrow\ 1+1/n \leq n^{\frac{\sqrt{n+1}}{\sqrt{n}}-1}.$$

By $(***)$ the latter inequality is satisfied if

$$(1+1/n)^n \leq n^{1/3}.$$

Since, as we proved in class, $(1+1/n)^n \leq 3$, it is enough to have

$$3^3 \leq n.$$

Thus the sequence $\{x_n\}_{n=27}^\infty$ is decreasing and bounded (by 27 above and by 1 below). Therefore it is convergent, which implies that $\{x_n\}_{n=1}^\infty$ is convergent. Now note that the sequence $\{y_n\}_{n=1}^\infty$, where $y_n = n^{1/n}$ is a subsequence of $\{x_n\}_{n=1}^\infty$. In class we proved that $y_n \to 1$ as $n \to \infty$. It implies that $x_n \to 1$ as $n \to \infty$. □

Problem 2. Find the domain and sketch the graph of the following functions

a. $|x-1|-|x+1|$, **b.** $\begin{cases} x, & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$

c. $\dfrac{x^4}{x^2}$ **d.** $\sqrt{1-x^2}$.

Solution. For graphs please see the attached file.
a. Let $f(x) = |x-1| - |x+1|$. Clearly dom$f = \mathbb{R}$, as the formula has sense for every real x. Now, by the definition of absolute value we have
i. for $x \geq 1$ $f(x) = (x-1) - (x+1) = -2$,
ii. for $-1 \leq x < 1$ $f(x) = -(x-1) - (x+1) = -2x$,
iii. for $x < -1$ $f(x) = -(x-1) + (x+1) = 2$.
b. Again, the formula has sense for every real x, so the domain is $\mathbb{R}$.
c. Here the formula has sense for every nonzero real number, so the domain is $\mathbb{R} \setminus \{0\}$. Note that for all nonzero x we have $x^4/x^2 = x^2$.
d. Since square root is defined for non-negative numbers only the domain here is the set of all x such that $1 - x^2 \geq 0$, which is equivalent to $-1 \leq x \leq 1$. □

Answer. **a.** $\mathbb{R}$, **b.** $\mathbb{R}$, **c.** $\mathbb{R} \setminus \{0\}$, **d.** $[-1, 1]$.

Problem 3. Does the limit of function f at the point a exist? If YES, find the limit and prove your answer using (ε/δ)-definition of the limit. If NO, prove it.

a. $a = 9,\ f(x) = \sqrt{x}$, **b.** $a = 1,\ f(x) = \sqrt{1-x^2}$,

c. $a = 2,\ f(x) = \begin{cases} 2x, & \text{if } x > 2, \\ 5, & \text{if } x = 2, \\ 6 - x, & \text{if } x < 2, \end{cases}$

d. $a = -1,\ f(x) = \begin{cases} |x|, & \text{if } x > -1, \\ x^2, & \text{if } x < -1, \end{cases}$ **e.** $a = 0,\ f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, \\ 1 - x, & \text{if } x < 0. \end{cases}$

Solution.
a. We show that the limit exists and equal to 3. First note that $(7, 11) \subset$ dom$f = [0, \infty)$, in other words our function is defined "near" a. Now fix an arbitrary $\varepsilon > 0$. Choose $\delta = \min\{1, \varepsilon\} > 0$. Assume that $|x - 9| < \delta$. Then, since $\delta \leq 11$ we have $x \in$ dom$f$ and

$$|f(x) - 3| = |\sqrt{x} - 3| = \frac{|x-9|}{|\sqrt{x}+3|} \leq \frac{\delta}{3} < \varepsilon.$$

Thus, by the definition,

$$\lim_{x \to 9} f(x) = 3.$$

b. Note that, as we found in Problem 2, dom$f = [-1, 1]$. In particular, for $x > 1$ the function is not defined. Thus, there is no γ such that $(1-\gamma, 1+\gamma) \setminus \{-1\} \subset$ dom$f = [-1, 1]$. By the definition, in such a case the limit does not exist.
c. We show that the limit exists and equal to 4. Note that the domain of f is $\mathbb{R}$, so the first condition in the definition of the limit (that f is defined "near" a) is satisfied. Now fix an arbitrary ε. Choose $\delta = \varepsilon/2$. Assume that $0 < |x - 2| < \delta$. Then we have $x \in$ dom$f$ and
if $2 < x < 2 + \delta$ then

$$|f(x) - 4| = |2x - 4| = 2|x - 2| < 2\delta = \varepsilon;$$

if $2-\delta < x < 2$ then

$$|f(x)-4| = |(6-x)-4| = |2-x| = |x-2| < \delta < \varepsilon.$$

Thus, by the definition,

$$\lim_{x\to 2} f(x) = 4.$$

d. We show that the limit exists and equal to 1. Note that the domain of f is $\mathbb{R}\setminus\{-1\}$, so the first condition in the definition of the limit (that f is defined "near" a) is satisfied. Now fix an arbitrary ε. Choose $\delta = \min\{1, \varepsilon/3\}$. Assume that $0 < |x-(-1)| = |x+1| < \delta$. Then we have $x \in \operatorname{dom} f$ and moreover, $x < 0$ (because $\delta \le 1$), which implies $|x| = -x$. Hence,
if $-1 < x < -1+\delta$ then

$$|f(x)-1| = ||x|-1| = |-x-1| = |x+1| < \delta \le \varepsilon/3 < \varepsilon;$$

if $-1-\delta < x < -1$ then $|x| \le 1+\delta \le 2$ and

$$|f(x)-1| = |x^2-1| = |x-1|\cdot|x+1| < \delta(|x|+1) \le 3\delta \le \varepsilon;$$

Thus, by the definition,

$$\lim_{x\to 1} f(x) = 1.$$

e. We show that limit does not exists.
Way 1. In (ε/δ)-language. Fix an arbitrary real number L. We show that L can not serve as a limit. We have to prove that there is $\varepsilon_0 > 0$ such that for every $\delta > 0$ one can find x_0 satisfying $0 < |x_0 - a| = |x_0| < \delta$ and $|f(x_0) - L| \ge \varepsilon_0$.

We choose $\varepsilon_0 = 1/4$ and consider two cases.
Case 1. $L \ge 1/2$.
Given an arbitrary $\delta > 0$ we choose $x_0 = \min\{1/16, \delta/2\}$. Then

$$0 < |x_0| = x_0 < \delta \qquad \text{and} \qquad |f(x_0) - L| = L - \sqrt{x_0} \ge \frac{1}{2} - \frac{1}{\sqrt{16}} = \frac{1}{4} = \varepsilon_0.$$

Case 2. $L < 1/2$.
Given an arbitrary $\delta > 0$ we choose $x_1 = -\delta/2$. Then

$$0 < |x_1| = -x_1 < \delta \qquad \text{and} \qquad |f(x_1) - L| = |1 - x_1 - L| = 1 - x_1 - L \ge 1 - L \ge 1/2 \ge \varepsilon_0.$$

It completes the proof.

Way 2. Using language of sequences and uniqueness of the limit. Assume that limit exists and denote it by L. Take $x_n = 1/n > 0$ (for all $n \ge 1$). Then for every n we have $x_n \ne 0$ and

$$\lim_{n\to\infty} x_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Therefore, by the definition of the limit, $L = 0$. On the other hand, consider $y_n = -1/n < 0$. Then for every n we have $y_n \ne 0$ and

$$\lim_{n\to\infty} y_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$$

Therefore, by the definition of the limit, $L = 1$. Since $1 \ne 0$, we obtain a contradiction. It implies that limit does not exists. □

Answer.
a. Yes, the limit is 3; **b.** No; **c.** Yes, the limit is 4; **d.** Yes, the limit is 1; **e.** No.