Solutions of Assignment # 1.

Problem 1. Let A, B, C be sets. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution. **Part 1.** First we show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Let $x \in A \cup (B \cap C)$. Then, by the definition, $x \in A$ or $x \in (B \cap C)$.

Case 1. If $x \in A$ then $x \in (A \cup B)$ and $x \in (A \cup C)$. Therefore $x \in (A \cup B) \cap (A \cup C)$.

Case 2. If $x \in (B \cap C)$ then $x \in B$ and $x \in C$. Therefore $x \in (A \cup B)$ and $x \in (A \cup C)$. It means that $x \in (A \cup B) \cap (A \cup C)$.

That proves that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Part 2. Now we show that $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Let $x \in (A \cup B) \cap (A \cup C)$. Then, by the definition, $x \in (A \cup B)$ and $x \in (A \cup C)$.

If $x \in A$ then clearly $x \in A \cup (B \cap C)$. Assume $x \notin A$. Since $x \in (A \cup B)$, we obtain that $x \in B$. Since $x \in (A \cup C)$, we obtain $x \in C$. It implies that $x \in (B \cap C)$. Therefore $x \in A \cup (B \cap C)$.

Thus we proved $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$

Part 1 and Part 2 prove the statement. □

Problem 2. Let P, Q, R be statements. Provide truth table for

a. $(P \text{ and } Q) \Rightarrow R$, **b.** $P \Rightarrow (Q \text{ or } R)$.

Solution. See the attached table. Note that $P_2 = (P \Rightarrow (Q \text{ or } R)) = ((\text{not } P) \text{ or } Q \text{ or } R)$. and $P_1 = ((P \text{ and } Q) \Rightarrow R) = ((\text{not } (P \text{ and } Q)) \text{ or } R) = ((\text{not } P) \text{ or } (\text{not } Q) \text{ or } R)$. □

Problem 3. Determine if the following statement is true or false. Write negations.

a. $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} \quad yx = 5$,

b. $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} \quad |x| = y^2 - 4y + 4$,

c. $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} \quad |y| = x^2 - 4x + 4$.

Solution.

a. The statement is false. Indeed, consider $x_0 = 0 \in \mathbb{Z}$. Then for every $y \in \mathbb{Z}$ we have $x_0 y = 0$, in other words there is no $y \in \mathbb{Z}$ such that $x_0 y = 5$. The negation is $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} \quad yx \neq 5$. (Note that in fact we proved that negation is true, i.e. there exists x, namely $x_0 = 0$, such that for every y we have $yx_0 \neq 5$)

b. The statement is false. Indeed, note that given x, if such an y exists then $|x| = (y-2)^2$, so either $y = \sqrt{|x|} + 2$ or $y = -\sqrt{|x|} + 2$. Now consider $x = 2$. Then either $y = \sqrt{2} + 2$ or $y = -\sqrt{2} + 2$. In both cases $y \notin \mathbb{Z}$, which shows that for $x = 2$ there is no $y \in \mathbb{Z}$ satisfying the equation. (Similar, but more precise argument would be: consider $x = 2$. Assume that there exists such an y, i.e. $2 = (y-2)^2$. Since $y \in \mathbb{Z}$, $y - 2 \in \mathbb{Z}$ so there exists a number $z \in \mathbb{Z}$ with $z^2 = 2$, which is false. Hence we got a contradiction.) The negation is $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} \quad |x| \neq y^2 - 4y + 4$.

c. The statement is false. Indeed, for every x consider $y_0 = x^2 - 4x + 5 = (x-2)^2 + 1$. Then $y_0 > 0$, so $|y_0| = y_0 = (x^2 - 4x + 4) + 1 > x^2 - 4x + 4$. Thus for every x we can find a y_0 which does not satisfy the equation. It means that there is no x such that for every y the equation holds, i.e. the statement is false. The negation is $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} \quad |y| \neq x^2 - 4x + 4$. (Note that in fact we proved that the negation is true). □

Remark. Note that the statement **b.** would be true if we had asked about a real number y, namely it is true that $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{R} \quad |x| = y^2 - 4y + 4$. Indeed, fix an arbitrary x and consider $y = \sqrt{|x|} + 2$. Then $y \in \mathbb{R}$ and $y^2 - 4y + 4 = (y-2)^2 = |x|$.