Quiz # 7

Problem 1. Provide the (ε/δ)-definition of limit of a function at a point.

Definition. Let f be a function $A \to \mathbb{R}$, $a, L \in \mathbb{R}$. Assume there exists $\gamma > 0$ such that $(a-\gamma, a+\gamma) \setminus \{a\} \subset A$. Assume also that

$$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon.$$

Then we say that L is the limit of f at a.

Remarks. 1. Note that the second condition can be written as

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in (a-\delta, a+\delta) \setminus \{a\} \qquad |f(x) - L| < \varepsilon.$$

2. Note that δ is function of ε (and only of ε).

Problem 2. Let $f(x) = x^2$. Does

$$\lim_{x \to 3} f(x)$$

exist? If YES find it and prove using the (ε/δ)-definition; if NOT explain why.

Solution. We show that

$$\lim_{x \to 3} f(x) = 9.$$

First note that the domain of f is $\mathbb{R}$, so there exists and interval around 3 in the domain. Now for a given arbitrary $\varepsilon > 0$ we choose $\delta = \min\{1, \varepsilon/7\}$. Assume $|x-3| \le \delta$. Then, using $\delta \le 1$, $|x+3| \le |x-3| + 6 \le \delta + 6 \le 7$. Therefore

$$|f(x) - 9| = |x^2 - 9| = |x-3||x+3| < 7\delta \le \varepsilon.$$

(in the last inequality we used $\delta \le \varepsilon/7$). So we proved that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x-3| < \delta$ implies $|f(x) - 9| < \varepsilon$. It completes the proof. $\square$

Answer.

$$\lim_{x \to 3} f(x) = 9.$$