Quiz # 3

Problem 1. Complete the sentence ...

Solution.
a. By the definition, in order to prove that $x = -a$, we have to check that $a + x = 0$.
b. By the definition, in order to prove that $x = a^{-1}$, we have to check that $a \cdot x = 1$. $\square$

Problem 2. Using only definitions, axioms, and 2 facts (given below), prove that
a. $1^{-1} = 1$; **b.** if $x \le y$ and $z \le 0$ then $y \cdot z \le x \cdot z$.

Solution.
a. We have to check that $1 \cdot 1 = 1$ (indeed, apply **Problem 1b** with $x = a = 1$). The equality $1 \cdot 1 = 1$ is true by the definition of 1 (recall 1 is the number w from axiom A7).
b. Assume $x \le y$ and $z \le 0$.
Way 1. By O5 we have $x + (-x) \le y + (-x)$. By the definition of $(-x)$ it implies $0 \le y + (-x)$. Applying O6 with $a = z$, $b = 0$, $c = y + (-x)$, we obtain $z \cdot (y + (-x)) \le 0 \cdot (y + (-x))$. By A9 and A1 it implies $z \cdot y + z \cdot (-x) \le (y + (-x)) \cdot 0$. Applying Facts 1 and 2, we have $z \cdot y + (-(z \cdot x)) \le 0$. Hence, by O5, $(z \cdot y + (-(z \cdot x))) + z \cdot x \le 0 + z \cdot x$. Applying A2 and A1, we get $z \cdot y + ((-(z \cdot x))) + z \cdot x) \le z \cdot x + 0$. By A1 and the definition of 0 (recall, 0 is the number θ from A3), we obtain $z \cdot y + (z \cdot x + (-(z \cdot x))) \le z \cdot x$. By the definition of $-(z \cdot x)$ it implies $z \cdot y + 0 \le z \cdot x$. Applying the definiton of 0 again, we obtain $z \cdot y \le z \cdot x$, which implies the desired result by A5.
Way 2. By O5 we have $z + (-z) \le 0 + (-z)$. By the definition of $(-z)$ and A1, it implies $0 \le (-z) + 0$. By the definition of 0, we get $0 \le (-z)$. Now we apply O6 with $a = x$, $b = y$, $c = -z$. We observe, $x \cdot (-z) \le y \cdot (-z)$. By Fact 2, $-(x \cdot z) \le -(y \cdot z)$. By O5, $-(x \cdot z) + x \cdot z \le -(y \cdot z) + x \cdot z$. By A1 it implies, $x \cdot z + (-(x \cdot z)) \le x \cdot z + (-(y \cdot z))$. By the definition of $-(x \cdot z)$, we have $0 \le x \cdot z + (-(y \cdot z))$. By O5, $0 + y \cdot z \le (x \cdot z + (-(y \cdot z))) + y \cdot z$. By A1 and A2, $y \cdot z + 0 \le x \cdot z + ((-(y \cdot z)) + y \cdot z)$. By A1 and the definition of 0, it implies $y \cdot z \le x \cdot z + (y \cdot z + (-(y \cdot z)))$. By the definition of $-(y \cdot z)$, we get $y \cdot z \le x \cdot z + 0$. The definition of 0 implies the desired result. $\square$