Quiz # 2

Problem 1. Let $A = \{1, 3\}$ and $B = \{-1, 1\}$. Determine if the following statement true or false. Provide explanations. Write the negation.
a. $\forall x \in A\ \exists y \in B \quad x + y = 2$; **b.** $\exists x \in A\ \forall y \in B \quad x + y \leq 2$.

Solution.
a. True. Indeed, if $x = 1$ we take $y = 1 \in B$, if $x = 3$ we take $y = -1 \in B$. In both cases $x + y = 2$, which proves that the statement is true. The negation is $\exists x \in A\ \forall y \in B \quad x + y \neq 2$.
b. True. Take $x = 1 \in A$. Then if $y = -1 \in B$ we have $x + y = 0 \leq 2$, if $y = 1 \in B$ we have $x + y = 2 \leq 2$. It shows that with $x = 1 \in A$ we have $\forall y \in B \quad x + y \leq 2$, which proves the result. The negation is $\forall x \in A\ \exists y \in B \quad x + y > 2$. □

Problem 2. Using the induction principle prove that for every $x > 0$ and every $n \in \mathbb{N}$ one has

$$(1 + x)^n \geq 1 + nx.$$

Solution.
1 (base). If $n = 1$ we have $(1 + x)^1 \geq 1 + 1 \cdot x$, which is clearly true (actually we have an equality).
2 (step). Assume the formula holds for n, that is $(1 + x)^n \geq 1 + nx$. Then

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + nx)(1 + x) = 1 + nx + x + nx^2 \geq 1 + (n + 1)x.$$

In the first inequality above we used the induction assumption and $x > 0$ (so $1 + x > 0$), in the second one we used that $nx^2 \geq 0$ for every $n \geq 1$ and x. This proves the induction step and, thus, concludes the proof of the general formula. □