Drill Problems # 7.

Problem 1. Let f be continuous at $a \in \mathbb{R}$. Show that $|f|$ is continuous at a.

Problem 2. Let $a > 0$ and $f : (-a, a) \to \mathbb{R}$ be such that for every $x \in (-a, a)$ one has $|f(x)| \leq |x|$. Show that $|f|$ is continuous at 0. Is f continuous at $\frac{a}{2}$?

Problem 3. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists $x \in [0, 1]$ such that $f(x) = x$.

Problem 4. Let $a < b$ and $f, g : [a, b] \to [a, b]$ be continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Prove that there exists $x \in [0, 1]$ such that $f(x) = x$.

Problem 5. Given that for every $x, y \in \mathbb{R}$ one has

$$|\sin x| \leq |x|, \qquad |\cos x| \leq 1, \qquad \sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2},$$

prove that the function sin is continuous.

Problem 6. Let $P(x)$ be a polynomial of odd degree. Prove that there exists $x \in \mathbb{R}$ such that $P(x) = 0$. Does the same hold for polynomials of even degree?

Problem 7. Is it true that for every $y \in \mathbb{R}$ and every polynomial $P(x)$ of odd degree there exists $x \in \mathbb{R}$ such that $P(x) = y$.

Problem 8. Let $f(x) = x$ if x is a rational number and $f(x) = 0$ if x is an irrational number. Find all points of continuity of f.

Problem 9. Let f be continuous on $(0, \infty)$. Assume that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to \infty} f(x) = \infty.$$

Prove that f attains its minimum.