Drill Problems # 4.

Problem 1. Let $\ell \in \mathbb{R}$, $c \in \mathbb{R}$, $c > 0$. Show that the following statements are equivalent.

a. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |x_n - \ell| < \varepsilon$;

b. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N \quad |x_n - \ell| < \varepsilon$;

c. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |x_n - \ell| \leq \varepsilon$;

d. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N \quad |x_n - \ell| \leq \varepsilon$;

e. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |x_n - \ell| < c\varepsilon$;

d. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |x_n - \ell| \leq c\varepsilon$;

Problem 2. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence and $b \in \mathbb{R}$. Is the following statement true or false?

a. If $\forall n \in \mathbb{N}\ x_n \geq b$ then $\lim\limits_{n\to\infty} x_n \geq b$.

b. If $\forall n \in \mathbb{N}\ x_n > b$ then $\lim\limits_{n\to\infty} x_n > b$.

As usual, you you claim "TRUE" then prove it; if you claim "FALSE" then provide a counterexample.

Problem 3. Let $x_n = \sqrt{n+1} - \sqrt{n}$ and $y_n = \sqrt{n}x_n$. Show that both sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are convergent and find limits.

Problem 4. Find limits (and then prove using only the definition) of the following sequences.

a. $\left\{\dfrac{1}{n^p}\right\}_{n=1}^{\infty}$, where $p > 0$ **b.** $\left\{\dfrac{(-1)^n}{\sqrt{n}+n^2}\right\}_{n=1}^{\infty}$, **c.** $\left\{\dfrac{(n+1)^2}{n^2}\right\}_{n=1}^{\infty}$.

Please note that you are not asked to find the BEST possible $N(\varepsilon)$ or, in your analysis, the BEST possible ε such that $|x_n - \ell| < \varepsilon$. Simplify your job, use trivial inequalities like

$$n \leq n^2, \quad \frac{1}{n^2} \leq \frac{1}{n}, \quad \frac{1}{n} + \frac{1}{n^2} \leq \frac{2}{n}, \quad \frac{1}{n+n^2} \leq \frac{1}{n}, \quad \frac{1}{n+8} \leq \frac{1}{n}, \ldots$$

Problem 5. Find limits of the following sequences (you can use facts from the class).

a. $\left\{\dfrac{5n^2+7n+8}{3n^2-2n+1}\right\}_{n=1}^{\infty}$, **b.** $\left\{\dfrac{2n^2-n+1}{n^3-n+1}\right\}_{n=1}^{\infty}$, **c.** $\left\{\dfrac{n\sqrt{n}+n-1}{n^2+\sqrt{n}}\right\}_{n=1}^{\infty}$.