Assignment # 6.

Due Oct. 23, 13:00

Problem 1. Let $x, y \in \mathbb{R}$ be such that $x \neq y$. Prove that $\exists \varepsilon > 0$ such that $y \notin [x - \varepsilon, x + \varepsilon]$.

Problem 2. Is it true that for every two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ satisfying $\forall n \;\; x_n < y_n$ one has

a. $\sup x_n \leq \sup y_n$? **b.** $\sup x_n < \sup y_n$?

Explain your answer (that is, prove if YES; provide a counterexample if NO).

Problem 3. A student was trying to recall the definition of a convergent sequence, and came up with the following statements. Your job is to persuade the student that these definitions are wrong, and not even equivalent to the correct definition. In order to do this, for each of these statements you need to find an example of a sequence which either satisfies the statement but fails to converge, or which converges but fails the statement (recall that a sequence $\{x_n\}_{n=1}^{\infty}$ is convergent to $\ell \in \mathbb{R}$ if $\forall \varepsilon > 0 \; \exists N \; \forall n \geq N \quad |x_n - \ell| < \varepsilon$).

a. $\forall \varepsilon \geq 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \quad |x_n - \ell| < \varepsilon$;

b. $\forall \varepsilon > 0 \; \exists n \in \mathbb{N} \quad |x_n - \ell| < \varepsilon$;

c. $\exists N \in \mathbb{N} \; \forall \varepsilon > 0 \; \forall n \geq N \quad |x_n - \ell| < \varepsilon$;

d. $\exists \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \quad |x_n - \ell| < \varepsilon$;

e. $\forall \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N \quad |x_n - \ell| < \varepsilon$.

Explain your answer.

Problem 4. Find limits (and then prove using the definition) of the following sequences.

a. $\left\{ \dfrac{1}{n^2} \right\}_{n=1}^{\infty}$, **b.** $\left\{ \dfrac{(-1)^n}{n} \right\}_{n=1}^{\infty}$, **c.** $\left\{ \dfrac{(n+2)^2}{n^2} \right\}_{n=1}^{\infty}$.