The extension of the finite-dimensional version of Krivine's theorem to quasi-normed spaces.

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Recently, a number of results of the Local Theory have been extended to the quasi-normed spaces. There are several works ([Kal1], [Kal2], [D], [GL], [KT], [GK], [BBP1], [BBP2], [M2]) where such results as Dvoretzky-Rogers lemma ([DvR]), Dvoretzky theorem ([Dv1], [Dv2]), Milman's subspace-quotient theorem ([M1]), Krivine's theorem ([Kr]), Pisier's abstract version of Grotendick's theorem ([P1], [P2]), Gluskin's theorem on Minkowski compactum ([G]), Milman's reverse Brunn-Minkowski inequality ([M3]), and Milman's isomorphic regularization theorem ([M4]) are extended to quasi-normed spaces after they were established for normed spaces. It is somewhat surprising since the first proofs of these facts substantially used convexity and duality.

In [AM2] D. Amir and V.D. Milman proved the local version of Krivine's theorem (see also [Gow], [MS]). They studied quantitative estimates appearing in this theorem. We extend their result to the q- and quasi-normed spaces.

Recall that the quasi-norm on a real vector space X is a map $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ such that

1) $\|x\| > 0 \quad \forall x \neq 0$,

2) $\|tx\| = |t| \, \|x\| \quad \forall t \in \mathrm{R},\ x \in X$,

3) $\exists\, C \geq 1$ such that $\forall x, y \in X \ \ \|x+y\| \leq C(\|x\| + \|y\|)$.

If 3) is substituted by

$3a)$ $\forall x, y \in X \ \|x+y\|^q \leq \|x\|^q + \|y\|^q$ for some fixed $q \in (0, 1]$

then $\|\cdot\|$ is called a q-norm on X. Note that 1-norm is the usual norm. It is obvious that every q-norm is a quasi-norm with $C = 2^{1/q-1}$. However, not every quasi-norm is q-norm for some q. Moreover, it is even not necessary

*This research was supported by Grant No.92-00285 from United States-Israel Binational Science Foundation (BSF).

continuous. It can be shown by the following simple example. Let f be a positive function on the Euclidean sphere S^{n-1} defined by

$$f(x) = \begin{cases} |x| & \text{for } x \in A, \\ 2|x| & \text{otherwise.} \end{cases}$$

Here A is a subset of S^{n-1} such that both A and $S^{n-1} \setminus A$ are dense in S^{n-1}. Denote $\|x\| = |x| f(x/|x|)$. Because f is not continuous it is clear that $\| \cdot \|$ is not q-norm for any q though it is the quasi-norm.

The next lemma is Aoki-Rolewicz Theorem ([KPR], [R], see also [K], p.47).

Lemma 1 *Let $\| \cdot \|$ be a quasi-norm with the constant C in the quasi-triangle inequality. Then there exists a q-norm $\| \cdot \|$ for which*

$$\|x\|_q \leq \|x\| \leq 2C\|x\|_q$$

with q satisfying $2^{1/q-1} = C$. *This q-norm can be defined as follows*

$$\|x\|_q = \inf \left\{ \left(\sum_{i=1}^{n} ||x_i||^q \right)^{1/q} : n > 0, \; x = \sum_{i=1}^{n} x_i \right\}.$$

We refer to [KPR] for further properties of the quasi- and q-norms.

Next, we prove the following theorem.

Theorem 1 *Let $\{e_i\}_1^n$ be a unit vector basis in $\mathbb{R}^n$, $\| \cdot \|_p$ be a l_p-norm on $\mathbb{R}^n$, i.e. $\| \sum_{i=1}^n a_i e_i \|_p = (\sum_i |a_i|^p)^{1/p}$, for $0 < p < \infty$. Let $\| \cdot \|$ be a q-norm on $\mathbb{R}^n$ such that*

$$C_1^{-1} \|x\|_p \leq \|x\| \leq C_2 \|x\|_p \qquad (*)$$

for every $x \in \mathbb{R}^n$. Then for every $\varepsilon > 0$ and $C = C_1 C_2$ there exists a block sequence $u_1, u_2, ..., u_m$ of $e_1, e_2, ..., e_n$ which satisfies

$$(1-\varepsilon) \left(\sum_{i=1}^{m} |a_i|^p \right)^{1/p} \leq \left\| \sum_{i=1}^{m} a_i u_i \right\| \leq (1+\varepsilon) \left(\sum_{i=1}^{m} |a_i|^p \right)^{1/p} \qquad (**)$$

for all $a_1, a_2, ..., a_m$ and $m \geq C(\varepsilon, p, q) \, (n / \log n)^{\nu}$, where

$$\nu = \frac{\alpha \varepsilon_0}{\varepsilon_0 + p + \alpha \varepsilon_0} \,, \text{ for } p < 1 \text{ and } \nu = \frac{\varepsilon_0}{2\varepsilon_0 + 1} \,, \text{ for } p \geq 1 \,;$$

$$\alpha = \min\{p, q\} \,, \; \varepsilon_0 = \left(\frac{q\varepsilon/2}{1 + C^q 12^{q/p}} \right)^{p/q} .$$

Remark 1. If $p \geq 1$ in this theorem, then we have the well-known finite-dimensional version of Krivine's theorem with some modifications concerning change of the usual norm to the q-norm. In this case for small enough q we get $\varepsilon_0 \approx \left(\frac{q\varepsilon}{4}\right)^{p/q}$ and $\nu \approx \varepsilon_0$.

The case $p < 1$ is more interesting. We get an extension of the finite-dimensional version of Krivine's theorem. To provide an intuition for the behavior of the constant in the theorem we point out that for small enough p and q with $p = q$ we can take $\varepsilon_0 \approx \frac{q\varepsilon}{30}$ and $\nu \approx \varepsilon_0$.

Remark 2. By Lemma 1 in the case of quasi-norm with the constant C_0 the inequality $(**)$ is substituted with

$$(1-\varepsilon)\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{m} a_i u_i\right\| \leq 2(1+\varepsilon)C_0\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p}.$$

Due to the example above, we can not remove the constant C_0 in this inequality.

The proof of the theorem consists of two lemmas.

Lemma 2 *For every $\eta > 0$ there exists a constant $C(\eta) > 0$ such that if $\|\cdot\|$ is a q-norm on $\mathbb{R}^n$ satisfying $(*)$ then there exists a block sequence $y_1, y_2, ..., y_k$ of $e_1, e_2, ..., e_n$ which is $(1+\eta)$-symmetric and $k \geq C(\eta, q, p)\,\frac{n}{\log n}$.*

Lemma 3 *If $y_1, y_2, ..., y_k$ is a 1-symmetric sequence in a normed space satisfying*

$$C_1^{-1}\|a\|_p \leq \left\|\sum_{i=1}^{k} a_i y_i\right\| \leq C_2\|a\|_p$$

for all $a = (a_1, a_2, ..., a_k) \in \mathbb{R}^k$ then for every $\varepsilon > 0$ there exists a block sequence $u_1, u_2, ..., u_m$ of $y_1, y_2, ..., y_k$ such that

$$(1-\varepsilon)\|a\|_p \leq \left\|\sum_{i=1}^{m} a_i u_i\right\| \leq (1+\varepsilon)\|a\|_p$$

for all $a = (a_1, a_2, ..., a_m) \in \mathbb{R}^m$, where $m \geq C(p,q)\varepsilon^{p/q}k^{\nu}$, $\nu = \frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}$, for $p < 1$ and $\nu = \frac{\varepsilon_0}{2\varepsilon_0+1}$, for $p \geq 1$, $\alpha = \min\{p, q\}$, $\varepsilon_0 = \left(\frac{q\varepsilon}{1+C^q 12^{q/p}}\right)^{p/q}$.

At first, D. Amir and V.D. Milman ([AM2], see also [MS]) proved Lemma 2 for $q=1$, $p \geq 1$ with the estimate $k \geq C(\eta,q,p)\, n^{1/3}$. Their proof can be modified to obtain result for $0<p<\infty$, $q \leq 1$. Afterwards, W.T. Gowers ([Gow]) showed that the estimate of k can be improved to $k \geq C(\eta,q,p) n/\ln n$. In fact, he gave two different, though similar, proofs for cases $p=1$ and $p>1$. The proof given for case $p=1$ strongly used the convexity of the norm and the fact that p is equal to 1. However, the method used for $p>1$ actually works for every $0<p<\infty$ and even for q-norms. Let us recall the idea of W.T. Gowers. First we will introduce some definition.

Let Ω be the group $\{-1,1\}^n \times S_n$, where S_n is the permutation group. Let Ψ be the group $\{-1,1\}^k \times S_k$. For

$$b = \sum_{i=1}^{n} b_i e_i \in I\!R^n, \quad a = \sum_{i=1}^{k} a_i e_i \in I\!R^k, \quad (\varepsilon,\pi) \in \Omega, \quad (\eta,\sigma) \in \Psi$$

denote

$$b_{\varepsilon\pi} = \sum_{i=1}^{n} \varepsilon_i b_i e_{\pi(i)}\,, \qquad a_{\eta\sigma} = \sum_{i=1}^{k} \eta_i a_i e_{\sigma(i)}\,.$$

Let $h \cdot k = n$. For $i \leq k$, $j \leq h$ put

$$e_{ij} = e_{(i-1)h+j}\,, \quad \varepsilon_{ij} = \varepsilon_{(i-1)h+j}\,, \quad \pi_{ij} = \pi((i-1)h+j).$$

Define an action of Ψ on Ω by

$$\Psi_{\eta\sigma}((\varepsilon,\pi)) = (\varepsilon^1,\pi^1), \ \text{ where } \ \varepsilon^1_{ij} = \eta_i \varepsilon_{\sigma(i)j}, \quad \pi^1_{ij} = \pi_{\sigma(i)j}.$$

For any $(\varepsilon,\pi) \in \Omega$ define the operator

$$\Phi_{\varepsilon\pi} \,:\, I\!R^k \longrightarrow I\!R^n \qquad \text{by} \qquad \Phi_{\varepsilon\pi}\left(\sum_{i=1}^{k} a_i e_i\right) = \sum_{i=1}^{k}\sum_{j=1}^{h} \varepsilon_{ij} a_i e_{\pi_{ij}}.$$

For every $a \in I\!R^k$ by M_a denote the median of $\Phi_{\varepsilon\pi}(a)$ taken over Ω. Finally, let $A = \{a \in l_p^k \,:\, \|a\|_p \leq 1, a_1 \geq a_2 \geq \ldots \geq a_k \geq 0\}$.

The following claim, which W.T. Gowers proved for case $p>1$ and $q=1$, is the main step in the proof of Lemma 2.

Claim 1 *Let $\|\cdot\|$ be a q-norm on $I\!R^n$ satisfying $\|x\|_p \leq \|x\| \leq B\|x\|_p$. There is a constant $C_0 = C(p,q,\delta,B)$ such that given $\lambda > 0$ for every $a \in A$*

$$\mathbf{Prob}_{\Omega}\left\{ \exists(\eta,\sigma) \,:\, \left|\, \|\Phi_{\varepsilon\pi}(a_{\eta\sigma})\|^q - M_a^q \,\right|^{1/q} > \frac{1}{2^{1/q}}\delta\|a\|_p h^{1/p} \right\} < 1/N$$

with $k = C_0 \frac{n}{\lambda \log n}$ and $N = k^{\lambda}$.

The proof of this claim can be equally well applied for all $0 < p < \infty$ and $0 < q \leq 1$. The only change that we have to do is to replace the triangle inequality

$$| \, \|x\| - \|y\| \, | \leq \|x - y\| \quad \text{by} \quad | \, \|x\|^q - \|y\|^q \, |^{1/q} \leq \|x - y\| \, .$$

The following two claims are technical and can be proved using ideas of [Gow] with small changes, connected with replacing $p \geq 1$ by $p < 1$ and the norm by q-norm.

Claim 2 *Let $0 < p < \infty$ and $\delta > 0$. There exist a constant λ, depending on p and δ only, such that for every integer k the set A contains a δ-net K of cardinality k^λ.*

Claim 3 *Let $|| \cdot ||$ be a q-norm on $\mathbb{R}^n$ satisfying $\|x\|_p \leq \|x\| \leq B\|x\|_p$. If there is $(\varepsilon, \pi) \in \Omega$ such that for every a in some δ-net K of A*

$$| \, \|\Phi_{\varepsilon\pi}(a_{\eta\sigma})\|^q - \|\Phi_{\varepsilon\pi}(a_{\eta_1\sigma_1})\|^q \, |^{1/q} \leq \delta \|a\|_p h^{1/p}$$

for every $(\eta, \sigma), (\eta_1, \sigma_1) \in \Psi$ then the block basis

$$\{\Phi_{\varepsilon\pi}(e_i)\}_{i=1}^k$$

of $(\mathbb{R}^k, || \cdot ||)$ is $(1 + 6\,(B\delta)^q)^{1/q}$-symmetric.

These three claims imply Lemma 2 in the standard way (see [Gow] for the details).

Proof of Lemma 3:
Our method of proof is close to the method used in [AM1], but our notation follows that of [MS] (ch. 10).

First, we will give the Krivine's construction of block basis. Let a and N be some integers which will be specified later. Let us introduce some set of numbers $\{\lambda_j\}_J$. We will say that set

$$\{B_{j,i}\}_{j \in J, i \in I}$$

(if *card* $I = 1$ then we have only one index j) is $\{\lambda_j\}_J$-set if
1) $B_{j,i} \subset \{1, ..., n\}$ for every $j \in J, i \in I$,

2) $B_{j,i}$ are mutually disjoint,
3) $card\ B_{j,i} = \lambda_j$ for every $j \in J,\ i \in I$.

Let us fix some $\{[\rho^j]\}$-set

$$\{A_{j,s}\}_{0 \le j \le N-1, 1 \le s \le m}$$

for $\rho = 1 + 1/a$.

For $0 \le j \le N-1$, $1 \le s \le m$ denote

$$Y_{j,s} = \sum_{i \in A_{j,s}} y_i$$

and define

$$z_s = \sum_{j=0}^{N-1} \rho^{(N-j)/p} Y_{j,s}.$$

Clearly, $\|z_1\| = \|z_2\| = ... = \|z_m\|$. The integer m will be defined from

$$k \approx m \sum_{j=0}^{N-1} [\rho^{(N-j)/p}] \approx m\rho^N (\rho - 1)^{-1} = ma \left(\frac{a+1}{a} \right)^N .$$

Finally, we define the block sequence $\{u_s\}_{s=1}^m$ by

$$u_s = z_s / \|z_s\|.$$

Now, as in [MS], we will establish the necessary estimates.

Fix $N, M \in \{T+1, T+2, ..., m\}$ and $t_s \in \{0, ..., T\}$ for $s \in \{1, ..., m\}$ such that

$$\sum_{s=1}^{M} \rho^{-t_s} = 1 + \eta,\ \ |\eta| = 1.$$

Then

$$\sum_{s=1}^{M} \rho^{-t_s/p} z_s = \sum_{s=1}^{M} \sum_{j=0}^{N-1} \rho^{(N-j-t_s)/p} Y_{j,s} =$$

$$= \sum_{i=0}^{N-1+T} \rho^{(N-i)/p} \sum_{s \le M,\, j \le N-1,\, j+t_s = i} \ \sum_{l \in A_{j,s}} y_l = \sum_{i=0}^{N-1+T} \rho^{(N-i)/p} \sum_{l \in B_i} y_l$$

for some $\{a_i\}$-set $\{B_i\}_{i=0}^{N-1+T}$, where

$$a_i = \sum_{s \le M,\, j \le N-1,\, j+t_s = i} [\rho^{i-t_s}],\ \ 0 \le i \le N-1+T\,.$$

Therefore, we can choose a vector z which has the same structure as z_s (i.e. $z = \sum_{j=0}^{N-1} \rho^{(N-j)/p} \sum_{i\in A_j} y_i$ for some $\{[\rho^j]\}$-set $\{A_j\}_{0\le j\le N-1}$) such that the difference Δ is

$$\Delta = \sum_{s=1}^{M} \rho^{-t_s/p} z_s - z = \sum_{s=1}^{N-1} \rho^{(N-i)/p} \sum_{l\in C_i} y_l + \sum_{s=N}^{N-1+T} \rho^{(N-i)/p} \sum_{l\in C_i} y_l$$

for some $\{b_j\}$-set $\{C_j\}_{i=0}^{N-1+T}$, where

$$b_j = \begin{cases} |[\rho^j - a_j]| & \text{for } 0 \le j \le N-1, \\ a_j & \text{for } N \le j \le N-1+T. \end{cases}$$

Using technique of [MS] (pp. 66-67) we obtain

$$\|\Delta\| \le C_2 \rho^{N/p} (4T + N|\eta| + NM\rho^{-T})^{1/p} \quad \text{and} \quad \|z\| \ge (1/C_1)\rho^{N/p}(N/2)^{1/p}.$$

Hence

$$\left| \left\| \sum_{s=1}^{M} \rho^{-t_s/p} u_s \right\|^q - 1 \right| \le \left\| \sum_{s=1}^{M} \rho^{-t_s/p} u_s - \frac{z}{\|z\|} \right\|^q =$$

$$= \left(\frac{\|\Delta\|}{\|z\|} \right)^q \le (C_1 C_2)^q \Big(\frac{8T}{N} + 2|\eta| + 2M\rho^{-T} \Big)^{q/p}.$$

Thus

$$\left| \left\| \sum_{s=1}^{M} \rho^{-t_s/p} u_s \right\|^q - 1 \right| \le C^q (12\varepsilon_0)^{q/p},$$

provided $T \le N\varepsilon_0$, $|\eta| \le \varepsilon_0$, $M\rho^{-T} \le m\rho^{-T} \le \varepsilon_0$, for some ε_0.
Assume $T = [N\varepsilon_0]$.

CASE 1. $p < 1$.

Let $\sum_{s=1}^m |\alpha_s|^p = 1$ and $a_s = |\alpha_s|$. Let $\alpha = \min\{p,q\}$ and $\delta = \varepsilon_0^{1/p}/m^{1/\alpha}$. Take $\beta_s = \rho^{-t_s/p}$ or $\beta_s = 0$, $t_s \in \{0,1,...,T\}$ such that $|a_s - \beta_s| \le \delta$ for every s. It is possible if $\rho^{-T/p} \le \delta$ and $1 - \rho^{-1/p} \le \delta$. Since $p \le 1$ it is enough to take a such that it satisfies following the inequalities

$$\Big(\frac{a}{a+1} \Big)^{[N\varepsilon_0]} \le \delta^p = \frac{\varepsilon_0}{m^{p/\alpha}} \quad \text{and} \quad \delta \ge \frac{1}{p(a+1)}.$$

Take $a = \left[\frac{1}{\delta p}\right] = \left[\frac{m^{1/\alpha}}{p\varepsilon_0^{1/p}}\right]$. Thus $\delta \ge \frac{1}{p(a+1)}$,

$$\Big| \sum \rho^{-t_s} - 1 \Big| = \Big| \sum \beta_s^p - 1 \Big| \le \Big| \sum (a_s + \delta)^p - 1 \Big| \le$$

$$\leq \left|\sum(a_s^p+\delta^p)-1\right| = \delta^p m \leq \varepsilon_0 \,,$$

and

$$\left| \ \left\|\sum_{s=1}^m \beta_s u_s\right\|^q - \left\|\sum_{s=1}^m \alpha_s u_s\right\|^q \right| \leq \left\|\sum_{s=1}^m |\beta_s - a_s| u_s\right\|^q \leq$$

$$\leq \delta^q \left\|\sum_{s=1}^m u_s\right\|^q \leq \delta^q m \leq \varepsilon_0^{q/p} \,.$$

Hence

$$\left| \ \left\|\sum_{s=1}^m \alpha_s u_s\right\|^q - 1\right| \leq \varepsilon_0^{q/p}(1 + C^q 12^{q/p}) \,,$$

if $m^{p/\alpha} \leq \varepsilon_0(\frac{1+a}{a})^{[N\varepsilon_0]}$ and $ma(\frac{1+a}{a})^N \leq k$, when $a = \left[\frac{m^{1/\alpha}}{p\varepsilon_0^{1/p}}\right]$. Choose N such that $(\frac{a}{1+a})^{N\varepsilon_0}$ is of the order $\varepsilon_0/m^{p/\alpha}$. Then

$$m \, \frac{m^{1/\alpha}}{p\varepsilon_0^{1/p}} \left(\frac{m^{p/\alpha}}{\varepsilon_0}\right)^{1/\varepsilon_0} = \frac{m^{1+1/\alpha+p/(\alpha\varepsilon_0)}}{\varepsilon_0^{1/p} p \varepsilon_0^{1/\varepsilon_0}} \sim k \,.$$

Thus, since $1/\alpha \geq \max\{1/p, 1/q\}$,

$$m \sim \varepsilon_0 \, (pk)^{\frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}} \sim \varepsilon_0 k^{\frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}}$$

and for $\varepsilon_1 = \varepsilon_0^{q/p}\left(1 + c^q 12^{q/p}\right)$

$$(1-\varepsilon_1)^{1/q} \left\|(\alpha_s)\right\|_p \leq \left\|\sum \alpha_s u_s\right\| \leq (1+\varepsilon_1)^{1/q} \left\|(\alpha_s)\right\|_p$$

holds. For ε_1 small enough ($\varepsilon_1 < 2^q - 1$) we obtain $1 - \varepsilon_1/q \leq (1-\varepsilon_1)^{1/q}$ and $1 + 2\varepsilon_1/q \geq (1+\varepsilon_1)^{1/q}$. Take $\varepsilon = 2\varepsilon_1/q$, then

$$\varepsilon_0 = \left(\frac{q\varepsilon/2}{1 + C^q 12^{q/p}}\right)^{p/q}$$

and

$$m \geq C(p,q)\varepsilon^{p/q} k^{\frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}} \,.$$

CASE 2. $p \geq 1$.

We use the same idea. Let $\sum_{s=1}^m |\alpha_s|^p = 1$ and $a_s = |\alpha_s|$. Let $\delta = \varepsilon_0/(C^p m)$. Take $\beta_s = \rho^{-t_s/p}$ or $\beta_s = 0$, $t_s \in \{0, 1, ..., T\}$ such that $|a_s^p - \beta_s^p| \leq$

δ for every s. It is possible if $\rho^{-T} \leq \delta$ and $1-\rho^{-1} \leq \delta$. These two conditions are met if

$$\left(\frac{a}{a+1}\right)^{[N\varepsilon_0]} \leq \delta = \frac{\varepsilon_0}{C^p m} \quad \text{and} \quad \delta \geq \frac{1}{a+1}\,.$$

Take $a = \left[\frac{1}{\delta}\right] = \left[\frac{C^p m}{\varepsilon_0}\right]$. Thus

$$\left|\sum \rho^{-t_s} - 1\right| = \left|\sum \beta_s^p - 1\right| \leq \left|\sum (a_s^p + \delta) - 1\right| = \delta m \leq \varepsilon_0.$$

Since

$$\left\|\sum_{s=1}^m u_s\right\| \leq C_1 C_2 \frac{\|\sum_{s=1}^m u_s\|_p}{\|z\|_p} \leq C_1 C_2 \Big(\frac{m \sum \rho^{N-j}[\rho^j]}{\|z\|_p^p}\Big)^{1/p} = C m^{1/p}$$

and

$$|\beta_s - a_s| \leq |\beta_s^p - a_s^p|^{1/p} \leq \delta^{1/p}\,,$$

we obtain

$$\left|\ \left\|\sum_{s=1}^m \beta_s u_s\right\|^q - \left\|\sum_{s=1}^m \alpha_s u_s\right\|^q\right| \leq \left\|\sum_{s=1}^m |\beta_s - a_s| u_s\right\|^q \leq$$

$$\leq \delta^{q/p} \|\sum_{s=1}^m u_s\|^q \leq \delta^{q/p} C^q m^{q/p} \leq \varepsilon_0^{q/p}\,.$$

Hence

$$\left|\ \left\|\sum_{s=1}^m \alpha_s u_s\right\|^q - 1\right| \leq \varepsilon_0^{q/p}(1 + C^q 12^{q/p})\,,$$

if $m \leq \frac{\varepsilon_0}{C^p}(\frac{1+a}{a})^{[N\varepsilon_0]}$ and $ma(\frac{1+a}{a})^N \leq k$, when $a = \left[\frac{C^p m}{\varepsilon_0}\right]$. Choose N such that $(\frac{a}{1+a})^{N\varepsilon_0}$ is of the order $\varepsilon_0/(C^p m)$. Then

$$m\,\frac{C^p m}{\varepsilon_0}\,\left(\frac{C^p m}{\varepsilon_0}\right)^{1/\varepsilon_0} = \left(\frac{C^p}{\varepsilon_0}\right)^{1+1/\varepsilon_0} m^{2+1/\varepsilon_0} \sim k\,.$$

Thus

$$m \geq \frac{\varepsilon_0}{C^p} k^{\frac{\varepsilon_0}{2\varepsilon_0+1}}$$

and for $\varepsilon_1 = \varepsilon_0^{q/p}\left(1 + C^q 12^{q/p}\right)$

$$(1-\varepsilon_1)^{1/q}\,\|(\alpha_s)\|_p \leq \left\|\sum \alpha_s u_s\right\| \leq (1+\varepsilon_1)^{1/q}\,\|(\alpha_s)\|_p$$

holds. For ε_1 small enough ($\varepsilon_1 < 2^q - 1$) we obtain $1 - \varepsilon_1/q \le (1-\varepsilon_1)^{1/q}$ and $1 + 2\varepsilon_1/q \ge (1+\varepsilon_1)^{1/q}$. Take $\varepsilon = 2\varepsilon_1/q$, then

$$\varepsilon_0 \;=\; \left(\frac{q\varepsilon/2}{1 + C^q 12^{q/p}}\right)^{p/q}$$

and

$$m \;\ge\; C(p,q)\varepsilon^{p/q} k^{\frac{\varepsilon_0}{2\varepsilon_0+1}} \,.$$

□

Acknowledgment. I want to thank Prof. V. Milman for his guidance and encouragement, and to thank the referee for his helpful remarks.

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