On the constant in the reverse Brunn-Minkowski inequality for p-convex balls.

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Abstract

This note is devoted to the study of the dependence on p of the constant in the reverse Brunn-Minkowski inequality for p-convex balls (i.e. p-convex symmetric bodies). We will show that this constant is estimated as

$$c^{1/p} \leq C(p) \leq C^{ln(2/p)/p},$$

for absolute constants $c > 1$ and $C > 1$.

Let $K \subset I\!R^n$ and $0 < p \leq 1$. K is called a p-convex set if for any $\lambda, \mu \in (0,1)$ such that $\lambda^p + \mu^p = 1$ and for any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. We will call a p-convex compact centrally symmetric body a p-ball.

Recall that a p-norm on real vector space X is a map $\| \cdot \| \ : X \longrightarrow I\!R^+$ such that

1) $\|x\| > 0 \quad \forall x \neq 0$,
2) $\|tx\| = |t| \, \|x\| \quad \forall t \in \mathrm{R},\ x \in X$,
3) $\forall x, y \in X \quad \|x + y\|^p \leq \|x\|^p + \|y\|^p$.

Note that the unit ball of p-normed space is a p-ball and, vice versa, the gauge of p-ball is a p-norm.

Recently, J. Bastero, J. Bernués, and A. Peña ([BBP]) extended the reverse Brunn-Minkowski inequality, which was discovered by V. Milman ([M]), to the class of p-convex balls. They proved the following theorem.

Theorem *Let* $0 < p \leq 1$. *There exists a constant* $C = C(p) \geq 1$ *such*

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that for all $n \geq 1$ and all p-balls $A_1, A_2 \subset I\!R^n$, there exists a linear operator $u : I\!R^n \longrightarrow I\!R^n$ with $|det(u)| = 1$ and

$$|uA_1 + A_2|^{1/n} \leq C\Big(|A_1|^{1/n} + |A_2|^{1/n}\Big), \qquad (*)$$

where $|A|$ denotes the volume of body A.

Their proof yields an estimate $C(p) \leq C^{ln(2/p)/p^2}$.

We will obtain a much better estimate for $C(p)$, namely

Theorem 1 *There exist absolute constants $c > 1$ and $C > 1$ such that the constant $C(p)$ in $(*)$ satisfies*

$$c^{1/p} \leq C(p) \leq C^{ln(2/p)/p}.$$

The proof of the Theorem ([BBP]) based on an estimate of the entropy numbers (see also [Pi]). We use the same idea, but obtain the better dependence of the constant on p.

Let us recall the definitions of the Kolmogorov and entropy numbers. Let $U : X \longrightarrow Y$ be an operator between two Banach spaces. Let $k > 0$ be an integer. The Kolmogorov numbers are defined by the following formula

$$d_k(U) = \inf \{ \|Q_S U\| \mid S \subset Y, \ \ \dim S = k \},$$

where $Q_S : Y \longrightarrow Y/S$ is a quotient map. For any subsets K_1, K_2 of Y denote by $N(K_1, K_2)$ the smallest number N such that there are N points $y_1, ..., y_N$ in Y such that

$$K_1 \subset \bigcup_{i=1}^{N} (y_i + K_2).$$

Denote the unit ball of the space X (Y) by B_X (B_Y) and define the entropy numbers by

$$e_k(U) = \inf \Big\{ \varepsilon > 0 \mid N(UB_X, \varepsilon B_Y) \leq 2^{k-1} \Big\}.$$

For p-convex balls ($0 < p \leq 1$) $B_1, B_2 \subset I\!R^n$ we will denote the identity operator $id : (I\!R^n, \|\cdot\|_1) \longrightarrow (I\!R^n, \|\cdot\|_2)$ by $B_1 \longrightarrow B_2$, where $\|\cdot\|_i$ ($i = 1, 2$) is the p-norm, whose unit ball is B_i.

Theorem 2 *Given $\alpha > 1/p - 1/2$, there exists a constant $C = C(\alpha, p)$ such that for any n and for any p-convex ball $B \subset R^n$ there exists an ellipsoid $D \subset \mathbb{R}^n$ such that for every $1 \leq k \leq n$*

$$max \; \{ \; d_k(D \longrightarrow B), \; e_k(B \longrightarrow D) \} \; \leq C(n/k)^{\alpha} \; .$$

Moreover, there is an absolute constant c such that

$$C(\alpha, p) \leq \left(\frac{2}{p}\right)^{c/p} \left(\frac{1}{1-\delta}\right)^{8/\delta} \;, \;\; for \;\;\; \alpha > \frac{3(1-p)}{2p} \;, \; \delta = \frac{3(1-p)}{2p\alpha} \;, \;\; p \leq 1/2 \qquad (**)$$

and

$$C(\alpha, p) \leq \left(\frac{2}{p}\right)^{c/p^2} \left(\frac{1}{1-\varepsilon}\right)^{\frac{2}{\varepsilon p^2}} \;, \;\; for \;\;\; \alpha > \frac{1}{p} - \frac{1}{2} \;, \; \varepsilon = \frac{1/p - 1/2}{\alpha}. \quad (***)$$

Remark 1. In fact, in [BBP] Theorem 2 was proved with estimate $(***)$. Using this result we prove estimate $(**)$.

In the following $C(\alpha, p)$ will denote the best possible constant from Theorem 2.

The main point of the proof is the following lemma.

Lemma 1 *Let $p, q, \theta \in (0, 1)$ such that $1/q - 1 = (1/p - 1)(1 - \theta)$ and $\gamma = \alpha(1 - \theta)$. Then*

$$C(\alpha, p) \leq 2^{1/p} 2^{1/(1-\theta)} (e/(1-\theta))^{\alpha} C_{p\theta}^{1/(1-\theta)} C(\gamma, q)^{1/(1-\theta)} \;,$$

where

$$C_{p\theta} = \frac{\Gamma(1 + (1-p)/p)}{\Gamma(1 + \theta(1-p)/p)\Gamma(1 + (1-\theta)(1-p)/p)} \;, \quad \Gamma \textit{ is the gamma-function.}$$

For reader's convenience we postpone the proof of this lemma.

Proof of Theorem 2: Take $q = 1/2, \; 1 - \theta = p/(1-p)$. Then $C_{p\theta} = (1-p)/p$ and, consequently, by Lemma 1,

$$C(\alpha, p) \leq c \left(\frac{e}{p}\right)^{\alpha} 2^{2/p} \left(\frac{1}{p}\right)^{1/p} C\left(\frac{\alpha p}{1-p} \; , 1/2\right).$$

Inequality $(***)$ implies

$$C\left(\frac{\alpha p}{1-p}\,,1/2\right) \le c\left(\frac{1}{1-\delta}\right)^{8/\delta} \quad , \quad \text{where} \ \ \delta = \frac{3(1-p)}{2p\alpha}\,.$$

Thus for $\alpha > 3(1-p)/(2p)$ and $p \le 1/2$ we obtain

$$C(\alpha, p) \le \left(\frac{2}{p}\right)^{c/p} \left(\frac{1}{1-\delta}\right)^{8/\delta}.$$

□

Proof of Theorem 1: By B. Carl's theorem ([C], or see Th. 5.2 of [Pi]) for any operator u between Banach spaces the following inequality holds

$$\sup_{k\le n} k^{\alpha} e_k(u) \ \le\ \rho_\alpha \sup_{k\le n} k^{\alpha} d_k(u)\,.$$

One can check that Carl's proof works in the p-convex case also and gives

$$\rho_\alpha \ \le\ C^{1/p}(C\alpha)^{C\alpha}$$

for some absolute constant C. Let us fix $\alpha = 2/p$. Then, by Theorem 2, we have that for any p-convex body K there exists an ellipsoid D such that

$$\max\{e_n(D \longrightarrow B),\ e_n(B \longrightarrow D)\} \ \le\ C^{ln(2/p)/p}\,.$$

The standard argument ([Pi]) gives the upper estimate for C_p.

To show the lower bound we use the following example. Let B_p^n be a unit ball in the space l_p^n and B_2^n be a unit ball in the space l_2^n. Denote

$$A = \frac{|B_2^n|^{1/n}}{|B_p^n|^{1/n}} = \frac{\Gamma(3/2)\Gamma^{1/n}(1+n/p)}{\Gamma^{1/n}(1+n/2)\Gamma(1+1/p)} \ge C_0 \frac{n^{1/p-1/2}}{\sqrt{1/p}}\,,$$

where C_0 is an absolute constant.

Consider a body

$$K = AB_p^n\,.$$

We are going to estimate from below

$$\frac{|\,UB_2^n + K\,|^{1/n}}{|\,UB_2^n\,|^{1/n} + |\,K\,|^{1/n}} = \frac{|\,UB_2^n + K\,|^{1/n}}{2|\,B_2^n\,|^{1/n}}$$

for arbitrary operator $U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $|det\, U| = 1$.

To simplify the sum of bodies in the example let us use the Steiner symmetrization with respect to vectors from the canonical basis of $\mathbb{R}^n$ (see, e.g., [BLM], for precise definitions). Usually the Steiner symmetrization is defined for convex bodies, but if we take the unit ball of l_p^n and any coordinate vector then we have the similar situation. The following properties of the Steiner symmetrization are well-known (and can be directly checked)
(i) it preserves the volume,
(ii) the symmetrization of sum of two bodies contains sum of symmetrizations of these bodies, and
(iii) given an ellipsoid UB_2^n, a consecutive application of the Steiner symmetrizations with respect to all vectors from the canonical basis results in the ellipsoid VB_2^n, where V is a diagonal operator (depending on U).

That means that in our example it is enough to consider a diagonal operator U with $|det\, U| = 1$.

Let $b \in (0, 1)$ and P_1 be the orthogonal projection on a coordinate subspace of dimension $n - 1$. Then direct computations give for every $r > 0$

$$|\, UB_2^n + rB_p^n \,| \geq 2 \int_0^{rb_p} |\, P_1 UB_2^n + brP_1 B_p^n \,| dx \geq 2rb_p |\, P_1 UB_2^n + brP_1 B_p^n \,|,$$

where $b_p = (p(1-b))^{1/p}$. Since $P_1 K = AB_p^{n-1}$, by induction arguments one has

$$|\, UB_2^n + K \,| \geq \left(2Ab^{(k-1)/2} b_p \right)^k |\, P_k UB_2^n + b^k P_k K \,|,$$

where P_k is the orthogonal projection on an arbitrary $n-k$-dimensional coordinate subspace of $\mathbb{R}^n$. Choosing $b = \exp\,(-\frac{2}{kp})$, P_k such that $|\, P_k UB_2^n \,| \geq |\, B_2^{n-k} \,|$ and $k = [n/2]$ we get

$$C(p) \geq \frac{|\, UB_2^n + K \,|^{1/n}}{2|\, B_2^n \,|^{1/n}} \geq \frac{1}{2} \left(2Ae^{-1/p} \, (2/k)^{1/p} \right)^{k/n} \left(\frac{|\, B_2^{n-k} \,|}{|\, B_2^n \,|} \right)^{1/n}$$

$$\geq c_1 \sqrt{p^{1/2} \, (4/e)^{1/p}}$$

for sufficiently large n and an absolute constant c_1. That gives the result for p small enough, i.e. $p \leq c_2$, where c_2 is an absolute constant. For $p \in (c_2, 1]$ the result follows from the convex case. $\square$

To prove Lemma 1 we will use the Lions-Peetre interpolation ([BL], [K]) with parameters $(\theta, 1)$.

Let us recall some definitions.

Let X be a quasi-normed space with an equivalent quasi-norms $\|\cdot\|_0$ and $\|\cdot\|_1$. Let $X_i = (X, \|\cdot\|_i)$.

Define $K(t,x) = \inf\{\|x_0\|_0 + \; t\|x_1\|_1 \mid x = x_0 + x_1\}$ and

$$\|x\|_{\theta,1} = \theta(1-\theta) \int_0^{+\infty} \frac{K(t,x)}{t^{1+\theta}} dt\,,$$

for $\theta \in (0,1)$.

The interpolation space $(X_0, X_1)_{\theta,1}$ is the space $(X, \|\cdot\|_{\theta,1})$.

Claim 1 *Let $\|\cdot\|_0 = \|\cdot\|_1 = \|\cdot\|$ be p-norms on space X. Then*

$$\frac{1}{C_{p\theta}}\|x\| \le \|x\|_{\theta,1} \le \|x\|$$

for every $x \in X$, with $C_{p\theta}$ as in Lemma 1.

Proof: $\|x\|_{\theta,1} \le \|x\|$ since

$$\inf\{\|x_0\|_0 + \; t\,\|x_1\|_1 \mid x = x_0 + x_1\} \le \min(1,t)\,\|x\|$$

and

$$\|x\|_{\theta,1} = \theta(1-\theta) \int_0^{+\infty} \frac{K(t,x)}{t^{1+\theta}} dt \le \theta(1-\theta) \int_0^{+\infty} \frac{\min(1,t)}{t^{1+\theta}} \|x\| dt = \|x\|.$$

By p-convexity of norm $\|\cdot\|$ for $a = \frac{\|y\|}{\|x\|} \le 1$ we have

$$\frac{\|y\| + t\|x-y\|}{\|x\|} \ge a + t(1-a^p)^{1/p} \ge \frac{t}{(1+t^s)^{1/s}}\,, \quad \text{where} \quad s = \frac{p}{1-p}.$$

Hence

$$K(t,x) = \inf\{\|x_0\|_0 + \; t\,\|x_1\|_1 \mid x = x_0 + x_1\} \ge \|x\|\,\frac{t}{(1+t^s)^{1/s}}$$

and

$$\frac{\|x\|_{\theta,1}}{\|x\|} \geq \theta(1-\theta)\int\limits_0^{+\infty}\frac{dt}{(1+t^s)^{1/s}t^\theta} = B\left(\frac{1-\theta}{s},\frac{\theta}{s}\right)\frac{\theta(1-s)}{s} =$$

$$= \frac{(\theta/s)\Gamma(\theta/s)\ ((1-\theta)/s)\Gamma((1-\theta)/s)}{(1/s)\Gamma(1/s)} = \frac{1}{C_{p\theta}}\,,$$

where $B(x,y)$ is the beta-function. This proves the claim. □

Claim 2 *Let $\|\cdot\|_0 = \|\cdot\|_1 = \|\cdot\|$ be norms on X. Then $\|x\|_{\theta,1} = \|x\|$ for every $x \in X$.*

Proof: In case of norm $K(t,x) = \min(1,t)\|x\|$. So, $\|x\|_{\theta,1} = \|x\|$. □

The next statement is standard (see [BL] or [K]).

Claim 3 *Let X_i, Y_i $(i=0,1)$ be quasi-normed spaces. Let $T : X_i \longrightarrow Y_i$ $(i=0,1)$ be a linear operator. Then*

$$\|T : (X_0,X_1)_{\theta,1} \longrightarrow (Y_0,Y_1)_{\theta,1}\| \leq \|T : X_0 \longrightarrow Y_0\|^{1-\theta}\|T : X_1 \longrightarrow Y_1\|^{\theta}.$$

Claim 4 *Let X_i $(i=0,1)$ be quasi-normed spaces. Then for every $N \geq 1$,*

$$\left(\ l_1^N\left(X_0\right), l_1^N\left(X_1\right)\right)_{\theta,1} = l_1^N\left((X_0,X_1)_{\theta,1}\right)$$

with equal norms.

Proof: The conclusion of this claim follows from equality

$$K(t, x=(x_1,x_2,...,x_N), l_1^N(X_0), l_1^N(X_1)) = \sum_{i=1}^N K(t,x_i,X_0,X_1)\,.$$

□

Claim 5 *Let X_i $(i=0,1)$ be quasi-normed spaces, Y be a p-normed space. Let $T : X_i$ $(i=0,1) \longrightarrow Y$ be a linear operator. Then for every $k_0, k_1 \geq 1$*

$$d_{k_0+k_1-1}(T : (X_0,X_1)_{\theta,1} \longrightarrow Y) \leq C_{p\theta}\, d_{k_0}^{1-\theta}(T : X_0 \longrightarrow Y) d_{k_1}^{\theta}(T : X_1 \longrightarrow Y).$$

Proof: As in convex case ([P]), fix $\varepsilon > 0$. Consider a subspace $S_i \subset Y$ $(i = 0, 1)$ such that $\dim S_i < k_i$ and

$$\|Q_{S_i}T : X_i \longrightarrow Y/S_i\| \leq (1+\varepsilon) d_{k_i}(T : X_i \longrightarrow Y).$$

Let $S = span(S_0, S_1) \subset Y$. Then $\dim S < k_0 + k_1 - 1$ and

$$\|Q_S T : X_i \longrightarrow Y/S\| \leq \|Q_{S_i}T : X_i \longrightarrow Y/S_i\|.$$

Note that quotient space of a p-normed space is again a p-normed one. Because of this, and by Claims 1 and 3,

$$\|Q_S T : (X_0, X_1)_{\theta,1} \longrightarrow Y/S\| \leq C_{p\theta}\|Q_S T : (X_0, X_1)_{\theta,1} \longrightarrow (Y/S, Y/S)_{\theta,1}\| \leq$$

$$\leq C_{p\theta}\|Q_S T : X_0 \longrightarrow Y/S\|^{1-\theta}\|Q_S T : X_1 \longrightarrow Y/S\|^{\theta} \leq$$
$$\leq C_{p\theta}\|Q_{S_0}T : X_0 \longrightarrow Y/S_0\|^{1-\theta}\|Q_{S_1}T : X_1 \longrightarrow Y/S_1\|^{\theta} \leq$$
$$\leq C_{p\theta}(1+\varepsilon)^2 d_{k_0}(T : X_0 \longrightarrow Y)^{1-\theta} d_{k_1}(T : X_1 \longrightarrow Y)^{\theta}.$$

This completes the proof. □

Proof of Lemma 1:
Step 1.

Let D be an optimal ellipsoid such that

$$d_k(D \longrightarrow B) \leq C(\alpha, p)(n/k)^{\alpha} \quad \text{and} \quad e_k(B \longrightarrow D) \leq C(\alpha, p)(n/k)^{\alpha}$$

for every $1 \leq k \leq n$.

Let $\lambda = C(\alpha, p)(n/k)^{\alpha}$.

Step 2.

Now denote the body $(B, D)_{\theta,1}$ by B_θ. By Claim 5 (applied for $k_0 = 1$), for every $1 \leq k \leq n$ we have

$$d_k(B_\theta \longrightarrow B) \leq C_{p\theta}\|B \longrightarrow B\|^{1-\theta}(d_k(D \longrightarrow B))^{\theta} \leq C_{p\theta}\lambda^{\theta}.$$

It follows from the definition of entropy numbers that B is covered by 2^{k-1} translates of λD with centers in $\mathbb{R}^n$. Replacing λD with $2\lambda D$ we can choose these centers in B. Therefore there are 2^{k-1} points $x_i \in B$ $(1 \leq i \leq 2^{k-1})$ such that

$$B \subset \bigcup_{i=1}^{2^{k-1}} (x_i + 2\lambda D).$$

This means that for any $z \in B$ there is some $x_i \in B$ such that $\|z - x_i\|_D \leq 2\lambda$. Also, by p-convexity, $\|z - x_i\|_B \leq 2^{1/p}$. By taking the operator $u_x : I\!R \longrightarrow X$, $u_x t = tx$ for some fixed x, and applying Claim 3 (or see [BL], [BS]) it is clear that

$$\|x\|_{B_\theta} \leq \|x\|_B^{1-\theta} \|x\|_D^\theta.$$

Hence, for any $z \in B$ there exists $x_i \in B$ such that

$$\|z - x_i\|_{B_\theta} \leq (2^{1/p})^{1-\theta} (2\lambda)^\theta ,$$

i.e.

$$e_k(B \longrightarrow B_\theta) \leq 2^{(1-\theta)/p} (2\lambda)^\theta.$$

Thus, we obtain

$$d_k(B_\theta \longrightarrow B) \leq C_{p\theta} \lambda^\theta \qquad \text{and} \qquad e_k(B \longrightarrow B_\theta) \leq 2^\theta 2^{(1-\theta)/p} \lambda^\theta$$

for every $1 \leq k \leq n$.

Step 3.

Lemma 2 *Let $B \subset I\!R^n$ be a p-convex ball and $D \subset I\!R^n$ be a convex body. Let $0 < \theta < 1$ and $B_\theta = (B, D)_{\theta,1}$. Then there exists a q-convex body B^q such that $B_\theta \subset B^q \subset 2^{1/q} B_\theta$, where $1/q - 1 = (1/p - 1)(1 - \theta)$.*

Proof : Take the operator $U : l_1^2(I\!R^n) \longrightarrow I\!R^n$ defined by $U((x, y)) = x + y$. Since

$$\|x + y\|_B \leq 2^{1/p-1} (\|x\|_B + \|y\|_B) \quad \text{and} \quad \|x + y\|_D \leq (\|x\|_D + \|y\|_D)$$

and by Claims 3, 4 we have

$$\|x + y\|_{B_\theta} \leq 2^{(1-\theta)(1/p-1)} \left(\|x\|_{B_\theta} + \|y\|_{B_\theta} \right).$$

But by the Aoki-Rolewicz theorem for every quasi-norm $\|\cdot\|$ with the constant C in the quasi-triangle inequality there exists a q-norm

$$\|\cdot\|_q = \inf\left\{\left(\sum_{i=1}^{n}\|x_i\|^q\right)^{1/q} \mid n>0,\ x=\sum_{i=1}^{n}x_i\right\}$$

such that $\|x\|_q \leq \|x\| \leq 2C\|x\|_q$ with q satisfying $2^{1/q-1} = C$ ([KPR], [R], see also [K], p.47).

Thus, $B_\theta \subset B^q \subset 2^{1/q}B_\theta$, where B^q is a unit ball of q-norm $\|\cdot\|_q$. □

Remark 2. Essentially, Lemma 2 goes back to Theorem 5.6.2 of [BL]. However, the particular case that we need is simpler and we are able to estimate the constant of equivalence.

Note that Lemma 2 can be easily extended to the more general case:

Lemma 2′ *Let $B_i \subset I\!\!R^n$ be a p_i-convex bodies for $i = 0, 1$ and $B_\theta = (B_0, B_1)_{\theta,1}$. Then there exists a q-convex body B^q such that $B_\theta \subset B^q \subset 2^{1/q}B_\theta$, where*

$$\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Remark 3. N. Kalton pointed out to us that the interpolation body $(B, D)_{\theta,1}$ between a p-convex B and an ellipsoid D is equivalent to some q-convex body for any $q \in (0, 1]$ satisfying

$$1/q - 1/2 > (1/p - 1/2)(1-\theta)\,.$$

To prove this result one have to use methods of [Kal] and [KT]. Certainly, with growing q the constant of equivalence becomes worse.

Step 4.

By definition of $C(\alpha, p)$ for B^q and $\gamma = \alpha(1-\theta)$ there exists an ellipsoid D_1 such that for every $1 \leq k \leq n$

$$d_k(D_1 \longrightarrow B^q) \leq C(\gamma, q)(n/k)^\gamma \ \text{ and } \ e_k(B^q \longrightarrow D_1) \leq C(\gamma, q)(n/k)^\gamma.$$

By the ideal property of the numbers d_k, e_k and because of the inclusion $B_\theta \subset B^q \subset 2^{1/q}B_\theta$, for every $1 \leq k \leq n$

$$d_k(D_1 \longrightarrow B_\theta) \leq 2^{1/q}C(\gamma, q)(n/k)^\gamma \ \text{ and } \ e_k(B_\theta \longrightarrow D_1) \leq C(\gamma, q)(n/k)^\gamma.$$

Step 5.

Let $a = 1 + [k(1-\theta)]$. Using multiplicative properties of the numbers $d_k,\ e_k$ we get

$$\begin{aligned} d_k(D_1 \longrightarrow B) &\leq d_{k+1-a}(D_1 \longrightarrow B_\theta) d_a(B_\theta \longrightarrow B) \\ &\leq C_{p\theta}\lambda^\theta 2^{1/q} C(\gamma, q)(n/k)^\gamma \left(\frac{1}{(1-\theta)^{1-\theta}\theta^\theta}\right)^\alpha \\ &\leq C(\alpha, p)^\theta \left(\frac{e}{1-\theta}\right)^{\alpha(1-\theta)} C_{p\theta} 2^{1/q} C(\gamma, q)(n/k)^\alpha \end{aligned}$$

and

$$\begin{aligned} e_k(B \longrightarrow D_1) &\leq e_{k+1-a}(B \longrightarrow B_\theta) e_a(B_\theta \longrightarrow D_1) \\ &\leq 2^\theta 2^{(1-\theta)/p} \lambda^\theta C(\gamma, q)(n/k)^\gamma \left(\frac{1}{(1-\theta)^{1-\theta}\theta^\theta}\right)^\alpha \\ &\leq C(\alpha, p)^\theta \left(\frac{e}{1-\theta}\right)^{\alpha(1-\theta)} 2^\theta 2^{(1-\theta)/p} C(\gamma, q)(n/k)^\alpha. \end{aligned}$$

By minimality of $C(\alpha, p)$ and since $1/q \leq 1 + (1-\theta)/p$ we have

$$C(\alpha, p) \leq C(\alpha, p)^\theta \left(\frac{e}{1-\theta}\right)^{\alpha(1-\theta)} C_{p\theta} 2^{1-\theta/p} 2C(\gamma, q)(n/k)^\alpha.$$

That proves Lemma 1. □

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