The covering numbers and "low M^*-estimate" for quasi-convex bodies. *

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Abstract

This article gives estimates on the covering numbers and diameters of random proportional sections and projections of quasi-convex bodies in $\mathbb{R}^n$. These results were known for the convex case and played an essential role in the development of the theory. Because duality relations cannot be applied in the quasi-convex setting, new ingredients were introduced that give new understanding for the convex case as well.

1. Introduction and notation.

Let $|\cdot|$ be a Euclidean norm on $\mathbb{R}^n$ and D be the ellipsoid associated to this norm. Denote

$$A(n,k) = \sqrt{\frac{n}{k}} \int\limits_{S^{n-1}} \sqrt{\sum_{i=1}^{k} x_i^2}\; d\sigma\,(x) = \frac{\sqrt{n}\;\Gamma\left(\frac{k+1}{2}\right)\;\Gamma\left(\frac{n}{2}\right)}{\sqrt{k}\;\Gamma\left(\frac{k}{2}\right)\;\Gamma\left(\frac{n+1}{2}\right)}\ ,$$

where σ is the normalized rotationally invariant measure on the Euclidean sphere S^{n-1} and $\Gamma\,(\cdot)$ is the Gamma-function. Then $A(n,k) < 1$ and $A(n,k) \longrightarrow 1$ as $n, k \longrightarrow \infty$. For any star-body K in $\mathbb{R}^n$ define $M_K = \int\limits_{S^{n-1}} \parallel x \parallel\, d\sigma(x)$, where $\parallel x \parallel$ is the gauge of K. Let M_K^* be M_{K^0}, where K^0 is the polar of K. For any

*This research was done while the authors visited MSRI; we thank the Institute for its hospitality.

†Research partially supported by BSF.

subsets K_1, K_2 of $\mathbb{R}^n$ denote by $N(K_1, K_2)$ the smallest number N such that there are N points $y_1, ..., y_N$ in K_1 such that

$$K_1 \subset \bigcup_{i=1}^{N} (y_i + K_2).$$

Recall that a body K is called quasi-convex if there is a constant c such that $K + K \subset cK$, and given a $p \in (0, 1]$ a body K is called p-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. Note that for the gauge $\| \cdot \| = \| \cdot \|_K$ associated with the quasi-convex (p-convex) body K the following inequality holds for any $x, y \in \mathbb{R}^n$:

$$\| x + y \| \leq c \max\{\| x \|, \| y \|\} \quad (\| x + y \|^p \leq \| x \|^p + \| y \|^p) \, .$$

In particular, every p-convex body K is also quasi-convex and $K + K \subset 2^{1/p} K$. A more delicate result is that for every quasi-convex body K ($K + K \subset cK$) there exists a q-convex body K_0 such that $K \subset K_0 \subset 2cK$, where $2^{1/q} = 2c$. This is the Aoki-Rolewicz theorem ([KPR], [R], see also [K], p.47). In this note by a body we always mean a compact star-body, i.e. a body K satisfying $tK \subset K$ for all $t \in [0, 1]$.

Let us recall the so-called "low M^*-estimate" result.

Theorem 1 *Let $\lambda \in (0, 1)$ and n be large enough. Let K be a centrally-symmetric convex body in $\mathbb{R}^n$ and $\| \cdot \|$ be the gauge of K. Then there exists a subspace E of $(\mathbb{R}^n, \| \cdot \|)$ such that $\dim E = [\lambda n]$ and for any $x \in E$ the following inequality holds*

$$\| x \| \geq \frac{f(\lambda)}{M_K^*} |x|$$

for some function $f(\lambda)$, $0 < \lambda < 1$.

Remark. An inequality of this type was first proved in [M1] with very poor dependence on λ and then improved in [M2] to $f(\lambda) = C(1 - \lambda)$. It was later shown ([PT]), that one can take $f(\lambda) = C\sqrt{1 - \lambda}$ (for different proofs see [M3] and [G]).

By duality this theorem is equivalent to the following theorem.

Theorem $1'$ *Let $\lambda \in (0, 1)$ and n be large enough. For every centrally-symmetric convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of rank $[\lambda n]$ such that*

$$PD \subset \frac{M_K}{f(\lambda)} PK.$$

Theorem 1 was one of the central ingredients in the proof of several recent results of Local Theory, e.g. the Quotient of Subspace Theorem ([M1]) and the Reverse Brunn-Minkowski inequality of the second name author (see, e.g. [MS] or [P]). Both these results were later extended to a p-normed setting in [GK] and [BBP]. The proofs have essentially used corresponding convex results and some kind of "interpolation". However, the main technical tool in the proof of these convex results, Theorem 1, was a purely "convex" statement. Let us also note an extension of Dvoretzky's theorem to the quasi-convex setting by Dilworth ([D]).

In this note we will extend Theorem 1 and Theorem 1′ to quasi-convex, not necessarily centrally-symmetric bodies. Since duality arguments cannot be applied to a non-convex body these two theorems become different statements. Also "M_K^*" should be substituted by an appropriate quantity not involving duality. Note that by avoiding the use of convexity assumption in fact we also simplified the proof for the convex case.

2. Main results.

The following theorem is an extension of Theorem 1′.

Theorem 2 *Let $\lambda \in (0,1)$ and n be large enough ($n > c/(1-\lambda)^2$). For any p-convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of rank $[\lambda n]$ such that*

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+1/p}} PK,$$

where $A_p = const^{\frac{\ln(2/p)}{p}}$.

Remark. To appreciate the strength of this inequality apply it to the standard simplex S inscribed in D. Then $M_S \approx \sqrt{n \cdot \log n}$ and therefore for every $\lambda < 1$ there are λn-dimensional projections containing a Euclidean ball of radius $f(\lambda)/\sqrt{n \cdot \log n}$. At the same time S contains only a ball of radius $1/n$. In fact, using this theorem for $S \cap rD$ for some special value r, we can eliminate the logarithmic factor and obtain the existence of λn-dimensional projections containing a Euclidean ball of radius $f_1(\lambda)/\sqrt{n}$. Another example is "p-convex simplex", S_p, defined for $p \in (0,1)$ as a p-convex hull of extremum points of S, i.e.

$$S_p = \left\{ \sum_{i=1}^{n+1} \lambda_i x_i \ ; \quad \lambda_i \geq 0 \ \text{ and } \ \sum_{i=1}^{n+1} \lambda_i^p \leq 1 \right\},$$

where $\{x_i\}_{i=1}^{n+1} = extr\, S$. Then Theorem 2 gives us the existence of λn-dimensional projections containing a Euclidean ball of radius $\frac{f(\lambda,p)}{n^{1/p}} \sqrt{\frac{n}{\log n}}$ however S_p contains only a ball of radius $1/n^{1/p}$.

The proof of Theorem 2 is based on the next three lemmas. The first one was proved by W.B. Johnson and J. Lindenstrauss in [JL]. The second one was proved in [PT] for centrally-symmetric convex bodies and is the dual form of the Sudakov's minoration theorem.

Lemma 1 *There is an absolute constant c such that if* $\varepsilon > \sqrt{c/k}$ *and* $N \leq 2e^{\varepsilon^2 k/c}$*, then for any set of points* $y_1, ..., y_N \in \mathbb{R}^n$ *and any orthogonal projection* P *of rank* k

$$\mu\Big(\{U \in O_n \mid \forall j : A(1-\varepsilon)\sqrt{k/n}\ |y_j| \leq |PUy_j| \leq A(1+\varepsilon)\sqrt{k/n}\ |y_j|\}\Big) > 0,$$

where μ *is the Haar probability measure on* O_n *and* $A = A(n,k) \in (1/2, 1)$

Lemma 2 *Let* K *be a body such that* $K + K \subset aK$*. Then*

$$N(D, tK) \leq 2e^{8n(aM_K/t)^2}.$$

Proof: M. Talagrand gave a direct simple proof of this lemma for the convex case ([LT], pp. 82-83). One can check that his proof does not use symmetry and convexity of the body and produces the estimate $N(D, tB) \leq 2e^{2n(aM_B/t)^2}$ for every body B, such that $B - B \subset aB$.

Now for a body K, satisfying $K + K \subset aK$ denote $B = K \cap -K$.

Then $B - B \subset aB$ and $M_B \leq 2M_K$, since

$$\| x \|_B = \max(\| x \|_K, \| x \|_{-K}) \leq \| x \|_K + \| x \|_{-K} .$$

Thus

$$N(D, tK) \leq N(D, tB) \leq 2e^{2n(2aM_K/t)^2}.$$

□

Lemma 3 *Let* B *be a body,* K *be a* p*-convex body,* $r \in (0,1)$*,* $\{x_i\} \subset rB$ *and* $B \subset \bigcup(x_i + K)$*. Then* $B \subset t_r K$*, where* $t_r = \frac{1}{(1-r^p)^{1/p}}$*.*

Proof: Let t_r be the smallest $t > 0$ for which $B \subset tK$. Then, obviously $t_r = \max\{\| x \|_K \mid x \in B\}$. Since $B \subset \bigcup(x_i + K)$, for any point x in B there are points x_0 in rB and y in K such that $x = x_0 + y$. Then by maximality of t_r and p-convexity of K we have $t_r^p \leq r^p t_r^p + 1$. That proves the lemma. □

Proof of Theorem 2:
Any p-convex body K satisfies $K + K \subset aK$ with $a = 2^{1/p}$. By Lemma 2 we have

$$N = N(D, tK) \leq 2 \cdot \exp\Big(2^{3+2/p} n (M_K/t)^2\Big),$$

i.e. there exist points $x_1, ..., x_N$ in D, such that

$$D \subset \bigcup_{i=1}^{N} (x_i + tK).$$

Denote $c_p = 2^{3+2/p}$. Let t and ε satisfy

$$c_p n \left(\frac{M_K}{t} \right)^2 \leq \frac{\varepsilon^2 k}{c}$$

and $\varepsilon > \sqrt{c/k}$ for c being the constant from Lemma 1.

Choose

$$\varepsilon = \frac{1 - \sqrt{\lambda}}{2\sqrt{\lambda}} \ .$$

Applying Lemma 1 we obtain that there exist an orthogonal projection P of rank k such that

$$PD \subset \bigcup (Px_i + tPK) \ \text{ and } \ \ |Px_i| \leq (1+\varepsilon)\sqrt{\frac{k}{n}} |x_i|.$$

Let $\lambda = k/n$. Denote $r = (1+\varepsilon)\sqrt{\lambda}$. Since K is p-convex Lemma 3 gives us

$$PD \subset t t_r PK \ \text{ for } \ t = \frac{\sqrt{c c_p} M_K}{\varepsilon \sqrt{\lambda}} \ \text{ and } \ \varepsilon^2 > \frac{c}{\lambda n} \ , \ r < 1.$$

Then for n large enough we get

$$PD \subset \frac{A_p M_K}{(1-\lambda)^{1+1/p}} PK,$$

for $A_p = const^{\frac{\ln(2/p)}{p}}$. This completes the proof. □

Theorem 2 can be also formulated in a global form.

Theorem $2'$ *Let K be a p-convex body in $\mathbb{R}^n$. Then there is an orthogonal operator U such that*

$$D \subset A_p' M_K (K + UK) \, ,$$

where $A_p' = const^{\frac{\ln(2/p)}{p}}$.

This theorem can be proved independently, but we show how it follows from Theorem 2.

Proof of Theorem 2′: First, let us assume that K is symmetric body. It follows from the proof of Theorem 2 that actually the measure of such projections is large. So we can choose two orthogonal subspaces E_1, E_2 of $\mathbb{R}^n$ such that $\dim E_1 = [n/2]$, $\dim E_2 = [(n+1)/2]$ and

$$P_i D \subset A_p^{''} M_K P_i K,$$

where P_i is the projection on the space E_i $(i = 1, 2)$. Denote by $I = id_{\mathbb{R}^n} = P_1 + P_2$ and $U = P_1 - P_2$. So $P_1 = \frac{I+U}{2}$ and $P_2 = \frac{I-U}{2}$. Then U is an orthogonal operator and for any $x \in D$ we have

$$x = P_1 x + P_2 x \subset \ A_p^{''} M_K \left(\frac{I+U}{2}\right) K + \ A_p^{''} M_K \left(\frac{I-U}{2}\right) K \subset$$

$$\subset A_p^{''} M_K \frac{K+K}{2} + A_p^{''} M_K \frac{UK - UK}{2} = A_p^{'} M_K (K + UK) \ .$$

That proves Theorem 2′ for symmetric bodies. In general case we need to apply the same trick as in the proof of Lemma 2. Denote $B = K \cap -K$. Then B is symmetric p-convex body so, by first part of the proof, there is an orthogonal operator U such that

$$D \subset A_p^{'} M_B (B + UB)\,,$$

Since $B \subset K$ and $M_B \le 2M_K$ (see proof of Lemma 2), we get the result. □

Let us complement Lemma 2 by mentioning how the covering number $N(K, tD)$ can be estimated. In the convex case this estimate is given by the Sudakov's inequality ([S]), in terms of the quantity M_K^*. More precisely, if K is a centrally-symmetric convex body, then

$$N(K, tD) \le 2e^{cn(M_K^*/t)^2} \ .$$

Of course, using duality for a non-convex setting leads to a weak result, and we suggest below a substitute for the quantity M_K^*.

For two quasi-convex bodies K, B define the following number

$$M(K, B) = \frac{1}{|K|} \int_K \| \, x \, \|_B \; dx,$$

where $|K|$ is the volume of K, and $\| \, x \, \|_B$ is the gauge of B. Such numbers are considered in [MP1], [MP2] and [BMMP].

Lemma 4 *Let K be a p-convex body and B be a body. Assume $B - B \subset aB$. Then*

$$N(K, tB) \le 2e^{(cn/p)(aM(K,B)/t)^p} \ ,$$

where c is an absolute constant.

Proof: We follow the idea of M. Talagrand of estimating the covering numbers in the case $K = D$ ([LT], pp. 82-83, see also [BLM] Proposition 4.2). Denote the gauge of K by $\| \cdot \|$ and the gauge of B by $| \cdot |_B$. Define the measure μ by

$$d\mu = \frac{1}{A} e^{-\|x\|^p} dx, \text{ where } A \text{ is chosen so that } \int\limits_{\mathbb{R}^n} d\mu = 1.$$

Let $L = \int\limits_{\mathbb{R}^n} |x|_B d\mu$. Then $\mu\{|x|_B \le 2L\} \ge 1/2$. Let $x_1, x_2, ...$ be a maximal set of points in K such that $|x_i - x_j|_B \ge t$. So the sets $x_i + \frac{t}{a} B$ have mutually disjoint interiors. Let $y_i = \frac{ab}{t} x_i$ for some b. Then, by p-convexity of K and convexity of the function e^t, we have

$$\mu\{y_i + bB\} = \frac{1}{A} \int\limits_{bB} e^{-\|x+y_i\|^p} dx \ge \frac{1}{A} \int\limits_{bB} e^{-(\|x\|^p + \|y_i\|^p)} dx =$$

$$= \frac{1}{A} e^{-\|y_i\|^p} \int\limits_{bB} e^{-\|x\|^p} dx \ge e^{-(ba/t)^p} \mu\{bB\}.$$

Choose $b = 2L$. Then $\mu\{bB\} \ge 1/2$ and, hence,

$$N(K, tB) \le 2e^{(2aL/t)^p} \ .$$

Now compute L. First, the normalization constant A is equal

$$A = \int\limits_{\mathbb{R}^n} e^{-\|x\|^p} dx = \int\limits_{\mathbb{R}^n} \int\limits_{\|x\|}^{\infty} (-e^{-t^p})' dt dx = \int\limits_0^{\infty} p t^{p-1} e^{-t^p} \int\limits_{\|x\| \le t} dx dt =$$

$$= \int\limits_{\|x\| \le 1} dx \int\limits_0^{\infty} p t^{p+n-1} e^{-t^p} dt = |K| \cdot \Gamma\left(1 + \frac{n}{p}\right),$$

where Γ is the gamma-function. The remaining integral is

$$\int\limits_{\mathbb{R}^n} |x|_B e^{-\|x\|^p} dx = \int\limits_{\mathbb{R}^n} |x|_B \int\limits_{\|x\|}^{\infty} (-e^{-t^p})' dt dx = \int\limits_0^{\infty} p t^{p-1} e^{-t^p} \int\limits_{\|x\| \le t} |x|_B dx dt =$$

$$= \int\limits_{\|x\|\le 1} |x|_B dx \int\limits_0^\infty p t^{p+n} e^{-t^p} dt = |K| \cdot M(K,B) \cdot \Gamma\left(1 + \frac{n+1}{p}\right) .$$

Using Stirling's formula we get

$$L \approx \left(\frac{n}{p}\right)^{1/p} M(K,B).$$

That proves the lemma. □

Remark. An analogous lemma for a p-smooth ($1 \le p \le 2$) body K and a convex centrally-symmetric body B was announced in [MP2]. Of course, the proof holds for all $p > 0$ and every quasi-convex centrally-symmetric body B. More precisely the following lemma holds.

Lemma 4′ *Let K and B be bodies. Let $B - B \subset aB$ and assume that for some $p > 0$ there is a constant c_p which depends only on p and the body K, such that*

$$\| x+y \|_K^p + \| x-y \|_K^p \le 2 \cdot (\| x \|_K^p + c_p \cdot \| y \|_K^p) \quad \textit{for all} \quad x, y \in \mathbb{R}^n.$$

Then

$$N(K, tB) \le 2e^{cn(c_p/p)(aM(K,B)/t)^p},$$

where c is an absolute constant.

Lemma 4′ is an extension of Lemma 2 in the symmetric case. Indeed, since Euclidean space is a 2-smooth space, then in the case where $K = D$ is an ellipsoid, we have $c_2(D) = 1$. By direct computation, $M(D,B) = \frac{n}{n+1} M_B$. Thus,

$$N(D, tB) \le 2e^{(cn)(M_B/t)^2} .$$

Define the following characteristic of K:

$$\tilde{M}_K = \frac{1}{|K|} \int\limits_K |x| dx ,$$

where $|\cdot| = |\cdot|_D$ is the Euclidean norm associated to D.

Lemma 4 shows that for p-convex body K

$$N(K, tD) \le 2e^{(cn/p)(2\tilde{M}_K/t)^p} .$$

Theorem 3 follows from this estimate by arguments similar to those in [MPi].

Theorem 3 *Let $\lambda \in (0,1)$ and n be large enough. Let K be a p-convex body in $\mathbb{R}^n$ and $\|\cdot\|$ be the gauge of K. Then there exists a subspace E of $(\mathbb{R}^n, \|\cdot\|)$ such that $\dim E = [\lambda n]$ and for any $x \in E$ the following inequality holds*

$$\| x \| \geq \frac{(1-\lambda)^{1/2+1/p}}{a_p \tilde{M}_K} |x| \,,$$

where a_p depends on p only (more precisely $a_p = const^{\frac{\ln(2/p)}{p}}$).

Proof: By Lemma 4 there are points $x_1, ..., x_N$ in K, such that $N < e^{c_p n (\tilde{M}_K/t)^p}$ and for any $x \in K$ there exists some x_i such that $|x - x_i| < t$. By Lemma 1 there exists an orthogonal projection P on a subspace of dimension $[\delta n]$ such that if t and ε satisfy

$$c_p n \left(\frac{\tilde{M}_K}{t} \right)^p < \frac{\varepsilon^2 \delta n}{c} \quad \text{and} \quad \varepsilon > \sqrt{\frac{c}{\delta n}}$$

we have

$$b|x_i| := (1-\varepsilon) A \sqrt{\delta} |x_i| \leq |P x_i| \leq (1+\varepsilon) A \sqrt{\delta} |x_i|$$

for every x_i. Let $E = \mathrm{Ker} P$. Then $\dim E = \lambda n$, where $\lambda = 1 - \delta$. Take x in $K \bigcap E$. There is x_i such that $|x - x_i| < t$. Hence

$$|x| \leq |x - x_i| + |x_i| \leq t + \frac{|P x_i|}{b} = t + \frac{|P(x - x_i)|}{b} \leq$$

$$\leq t + \frac{|x - x_i|}{b} \leq t(1 + \frac{1}{b}) \leq \frac{const \cdot t}{(1-\varepsilon)\sqrt{\delta}} \,.$$

Therefore for n large enough and

$$t = \left(\frac{const \cdot c_p}{\varepsilon^2 \delta} \right)^{1/p} \tilde{M}_K$$

we get

$$\| x \| \geq \frac{const \cdot \varepsilon^2 (1-\varepsilon) \delta^{1/2+1/p}}{c_p^{1/p} \tilde{M}_K} |x| \,.$$

To obtain our result take ε, say, equal to 1/2. □

As was noted in [MP2] in some cases $\tilde{M}_K << M_K^*$ and then Theorem 3 gives better estimates than Theorem 1 even for a convex body (in some range of λ). As an example, if $K = B(l_1^n)$, then $\tilde{M}_K \leq c \cdot n^{-1/2}$, but $M_K^* \geq c \cdot n^{-1/2} (\log n)^{1/2}$ for some absolute constant c.

3. Additional remarks.

In fact, the proof of Theorem 2 shows a more general fact.

Fact. *Let D be an ellipsoid and K be a p-convex body. Let*

$$N(D, K) \le e^{\alpha n}.$$

For an integer $1 \le k \le n$ write $\lambda = k/n$. Then for some absolute constant c and

$$\gamma = c\sqrt{\alpha}, \quad k \in (\gamma^2 n, (1-2\gamma)^2 n)$$

there exists an orthogonal projection P of rank k such that

$$c_1 \left(p(1-\sqrt{\lambda})/2\right)^{1/p} PD \subset PK,$$

where c_1 is an absolute constant.

In terms of entropy numbers this means that

$$c_1 \frac{\left(p(1-\sqrt{k/n})/2\right)^{1/p}}{e_k(D,K)} PD \subset PK,$$

where $e_k(D, K) = \inf\{\varepsilon > 0 \mid N(D, \varepsilon K) \le 2^{k-1}\}$.

It is worthwhile to point out that Theorem 2 can be obtained from this result.

We thank E. Gluskin for his remarks on the first draft of this note.

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