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QUOTIENTS OF FINITE-DIMENSIONAL QUASI-NORMED SPACES

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Abstract. We study the existence of cubic quotients of finite-dimensional quasi-normed spaces, that is, quotients well isomorphic to ℓ_∞^k for some k. We give two results of this nature. The first guarantees a proportional dimensional cubic quotient when the envelope is cubic; the second gives an estimate for the size of a cubic quotient in terms of a measure of non-convexity of the quasi-norm.

1. Introduction

It is by now well-established that many of the core results in the local theory of Banach spaces extend to quasi-normed spaces (cf. [2], [3], [4], [7], [8], [9], [10], [13], [15], [16], [17] for example). In this note we give two results on the local theory of quasi-normed spaces which are of interest only in the non-convex situation.

Let us introduce some notation. Let X be a real finite-dimensional vector space. Then a p-norm $\|\cdot\|$ on X, $p \in (0,1]$, is a map $x \mapsto \|x\|$ $(X \mapsto \mathbb{R})$ so that:
(i) $\|x\| > 0$ if and only if $x \neq 0$.
(ii) $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$.
(iii) $\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p$ for $x_1, x_2 \in X$.
Then $(X, \|\cdot\|)$ is called a p-normed space. For the purposes of this paper a quasi-normed space is always assumed to be a p-normed space for some $p \in (0,1]$ (note that by Aoki-Rolewicz theorem on quasi-normed space one can introduce an equivalent p-norm ([12], [14], [21])). The set $B_X = \{x : \|x\| \leq 1\}$ is the unit ball of X. The closed convex hull of B_X, denoted by $\hat{B}_X$, is the unit ball of a norm

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$\|\cdot\|_{\hat{X}}$ on X; the corresponding normed space, $\hat{X}$, is called the Banach envelope of X.

The set B is called p-convex if for every x, $y \in B$ and every positive λ, μ satisfying $\lambda^p + \mu^p = 1$ one has $\lambda x + \mu y \in B$. Clearly, the unit ball of p-normed space is a p-convex set and, vise versa, a closed centrally-symmetric p-convex set is the unit ball of some p-norm provided that it is bounded and 0 belongs to its interior.

If X and Y are p-normed spaces (for some p) then the Banach-Mazur distance $d(X,Y)$ is defined as $\inf\{\|T\|\|T^{-1}\|\}$, where the infimum is taken over all linear isomorphisms $T : X \to Y$. We let $d_{B_X} = d_X = d(X, \ell_2^{\dim X})$ and $\delta_{B_X} = \delta_X = d(X, \hat{X})$. It is clear that δ_X is measure of non-convexity; in fact $\delta_X = \inf\{d(X,Y) :$ Y is a Banach space$\}$.

We now describe our main results. In Section 3 we investigate quasi-normed spaces X such that $\hat{X}$ satisfies an estimate $d(\hat{X}, \ell_\infty^{\dim X}) \le C$. It has been known for some time that non-trivial examples of this phenomenon exist [11]. In geometrical terms this means that the convex hull of the unit ball of X is close to a cube. We show using combinatorial results of Alesker, Szarek and Talagrand [1], [20] based on the Sauer-Shelah Theorem [18], [19] that X then has a proportional dimensional quotient E satisfying an estimate $d(E, \ell_\infty^{\dim E}) \le C'$. A much more precise statement is given in Theorem 3.4. We then use this result in Section 4 to prove that a p-normed space X has a quotient E with $\dim E \ge c_p \ln \delta_X / (\ln\ln \delta_X)$ and $d(E, \ell_\infty^{\dim E}) \le C_p$ where $0 < c_p, C_p < \infty$ are constants depending on p only. Again a more precise statement is given in Theorem 4.2.

In developing these results, we found it helpful to use the notion of a geometric hull of a subset of $\mathbb{R}^n$. Thus instead of considering a p-convex set B_X we consider an arbitrary compact spanning set S and then compare the absolutely convex hull ΔS with certain subsets $\Gamma_\theta S$ which can be obtained from S by geometrically converging series. Precisely $x \in \Gamma_\theta S$, $\theta \in (0,1)$, if and only if $x = (1-\theta)\sum_{n=0}^\infty \theta^n \lambda_n s_n$ where $s_n \in S$ and $|\lambda_n| \le 1$. Note that $\Gamma_\alpha \subset \frac{1-\alpha}{1-\theta}\Gamma_\theta$ for every $0 < \alpha < \theta < 1$. Our results can be stated in terms of estimates for the speed of convergence of $\Gamma_\theta S$ to ΔS as $\theta \to 1$. In this way we can derive results which are independent of $0 < p < 1$ and then obtain results about p-normed spaces as simple Corollaries. We develop the idea of the geometric hull in Section 2 and illustrate it by restating the quotient form of Dvoretzky's theorem in this language.

2. Approximation of convex sets

Let S be a subset of $\mathbb{R}^n$. Denote by ΔS the absolutely convex hull of S and by $\tilde{S}$ the star-shaped hull of S, i.e. $\tilde{S} = \{\lambda x : \ |\lambda| \le 1, \ x \in S\}$. For each $m \in \mathbb{N}$ we define $\Delta_m S$ to be the set of all vectors of the form $\frac{1}{m}(\lambda_1 x_1 + \cdots + \lambda_m x_m)$ where $|\lambda_k| \le 1$ and $x_k \in S$ for $1 \le k \le m$. If $0 < \theta < 1$ we define *the θ-geometric hull* of S, $\Gamma_\theta S$ to be the set of all vectors of the form $(1-\theta)\sum_{k=0}^{\infty} \lambda_k x_k$ where $|\lambda_k| \le \theta^k$ and $x_k \in S$ for $k = 0, 1, \cdots$.

Lemma 2.1. *Let S be a p-convex closed set where $0 < p < 1$. Then for $0 < \theta < 1$ we have*

$$\Gamma_\theta S \subset \left(p^{-1/p}(1-\theta)^{1-1/p}\right) S.$$

Proof. This follows easily from:

$$\frac{1-\theta}{(1-\theta^p)^{1/p}} \le p^{-1/p}(1-\theta)^{1-1/p}$$

which in turns from the estimate

$$\theta^p \le 1 - p(1-\theta).$$

□

Lemma 2.2. *If $\frac{1}{3} < \theta < 1$ and $m \in \mathbb{N}$ then*

$$\Gamma_\theta \Delta_m S \subset \frac{2\theta}{3\theta - 1} \Gamma_{\theta^{\frac{1}{m}}} S.$$

Proof. Note that

$$\Delta_m S \subset \frac{1}{m} \theta^{\frac{1}{m}-1} \sum_{k=0}^{m-1} \theta^{\frac{k}{m}} \tilde{S}.$$

Hence

$$\Gamma_\theta \Delta_m S \subset \frac{1-\theta}{m(1-\theta^{\frac{1}{m}})} \theta^{\frac{1}{m}-1} \Gamma_{\theta^{\frac{1}{m}}} S.$$

Now observe

$$\begin{aligned} \frac{1-\theta}{m(1-\theta^{\frac{1}{m}})}\theta^{\frac{1}{m}-1} &= \frac{\theta^{-1}-1}{m(\theta^{-\frac{1}{m}}-1)} \\ &\le \frac{\theta^{-1}-1}{|\ln\theta|} \\ &\le \frac{2}{3-\theta^{-1}}. \end{aligned} \tag{2.1}$$

This completes the proof. □

In this section, we make a few simple observations on the geometric hulls $\Gamma_\theta S$. Let us suppose that S is compact and spanning so that ΔS coincides with the unit ball B_X of a Banach space X, $\|\cdot\|_X$. Given $q \in [1,2]$ let $T_q = T_q(X)$ denote the equal-norm type q constant, i.e. the smallest constant satisfying

$$\operatorname*{Ave}_{\epsilon_k=\pm 1} \left\| \sum_{k=1}^N \epsilon_k x_k \right\|_X \le T_q N^{1/q} \max_{1\le k\le N} \|x_k\|$$

for every N. Given an integer N let b_N denote the least constant so that

$$\inf_{\epsilon_k=\pm 1} \left\| \sum_{k=1}^N \epsilon_k x_k \right\|_X \le b_N N \max_{1\le k\le N} \|x_k\|.$$

Given a set A by $|A|$ we denote the cardinality of A.

The following Lemma abstracts the idea of [7], Lemma 2.

Lemma 2.3. *Suppose* $\frac{1}{3} < \theta < 1$*, and let* $m = m(S)$ *be an integer such that* $\sum_{k=1}^\infty b_{2^k m} \le \theta$*. Then*

$$\Delta S \subset \frac{2\theta}{(3\theta-1)(1-\theta)} \Gamma_{\theta^{\frac{1}{m}}} S.$$

Proof. Suppose $N \in \mathbb{N}$ and suppose $u \in \Delta_{2N} S$. Then $u = \frac{1}{2N}(x_1 + \cdots + x_{2N})$ where $x_k \in \tilde{S}$. Hence there is a choice of signs $\epsilon_k = \pm 1$ with $|\{\epsilon_k = -1\}| \le N$ and

$$\left\| \sum_{k=1}^{2N} \epsilon_k x_k \right\|_X \le 2N b_{2N}.$$

Let $v = \frac{1}{N}(\sum_{\epsilon_k=1} x_k)$. Then $\|u - v\|_X \le b_{2N}$. Hence $\Delta_{2N} S \subset \Delta_N S + b_{2N} \Delta S$. Iterating we get

$$\Delta_{2^k m} S \subset \Delta_m S + \sum_{j=1}^k b_{2^j m} \Delta S$$

which leads to

$$\Delta S \subset \Delta_m S + \theta \Delta S$$

which implies

$$\Delta S \subset (1-\theta)^{-1} \Gamma_\theta \Delta_m S \subset \frac{2\theta}{(3\theta-1)(1-\theta)} \Gamma_{\theta^{\frac{1}{m}}} S.$$

□

Proposition 2.4. *(i) Suppose $1 < q \leq 2$ and q' be such that $1/q + 1/q' = 1$. Then for*

$$\theta = 1 - \frac{1}{4}\left(\frac{2^{1/q'} - 1}{2T_q}\right)^{q'}$$

we have $\Delta S \subset 12\Gamma_\theta S$.
(ii) There exists constant $C < \infty$ so that if m is the largest integer such that X has a subspace Y of dimension m with $d(Y, \ell_1^m) \leq 2$ then $\Delta S \subset 8\Gamma_\theta S$ for

$$\theta = 1 - \frac{1}{2}\,(Cm)^{-C\log\log(Cm)}\,.$$

Remark. We conjecture that the sharp estimate in (ii) is $\theta = 1 - c/m$.

Proof. (i) Observe that $b_N \leq T_q N^{\frac{1}{q}-1}$. Hence

$$\sum_{k=1}^{\infty} b_{2^k N} \leq T_q N^{-\frac{1}{q'}}(2^{\frac{1}{q'}} - 1)^{-1}.$$

Let N be the largest integer so that the right-hand side is at most $\frac{1}{2}$. Applying Lemma 2.3 with $\theta_0 = 1/2$ we obtain

$$\Delta S \subset 4\Gamma_{2^{-1/N}} S.$$

The result follows, since

$$\frac{1}{N} \leq \left(\frac{2^{1/q'} - 1}{2T_q}\right)^{q'} \leq \frac{1}{N-1} \quad \text{and} \quad \Gamma_\alpha \subset \frac{1-\alpha}{1-\theta}\Gamma_\theta$$

for $\alpha < \theta$.

In (ii) we note first by a result of Elton [5] (see also [22] for a sharper version) there exist universal constants $1/2 \leq c_0 < 1$ and $C \geq 1$ so that $b_{N_0} < c_0$ for some $N_0 \leq Cm$.

Recall simple properties of the numbers b_k. Clearly, for every k, l one has $b_{kl} \leq b_k b_l$ and $(k+l)b_{k+l} \leq k b_k + l b_l$. Thus if $b_k \leq c_0 < 1$ then $b_l \leq c = (1+c_0)/2 < 1$ for every $k \leq l \leq 2k$. Therefore we may suppose that N_0 is a power of two, say $N_0 = 2^q$, $q \geq 1$, and $b_{N_0} \leq c < 1$. Since $b_l \leq 1$ for every l, we get $b_{N_0^s l} \leq c^s$ for every integers $s \geq 1$, $l \geq 0$. Then, taking $N = N_0^r$ for some $r \geq 1$ we have

$$\sum_{k=1}^{\infty} b_{2^k N} = \sum_{j=0}^{\infty}\sum_{l=1}^{rq} b_{2^{rq+jrq+l}} \leq rq \sum_{j=1}^{\infty} c^{jr} \leq 2rqc^r \leq 1/2$$

provided $r \geq c_1 \ln q$ with appropriate absolute constant c_1.

Now take r to be smallest integer larger than $c_1 \ln q = c_1 \ln \log_2 N_0$. Then by Lemma 2.3 we obtain

$$\Delta S \subset 4\Gamma_{2^{-1/N}} S$$

for $N \sim (C'm)^{C' \log\log(C'm)}$ and the result follows. □

Corollary 2.5. *There are absolute constants c, $C > 0$ so that if X is a p-normed space then there exists a subspace Y in the envelope $\hat{X}$ such that dimension of Y is*

$$m \geq cp \exp\left\{ \frac{\ln A}{\ln \ln A} \right\},$$

where $A = C(\delta_X)^{p/(1-p)}$, and

$$d\left(Y, \ell_1^m\right) \leq 2.$$

Proof. Let $S = B_X$ and let m be as in Proposition 2.4. Then by the proposition we have $\Delta B_X \subset 8\Gamma_\theta B_X$ with

$$\theta = 1 - \frac{1}{2}\left(Cm\right)^{-C \log\log(Cm)}.$$

Thus by Lemma 2.1 we obtain

$$\Delta B_X \subset 8p^{-1/p} 2^{-1+1/p} \left(Cm\right)^{-(1-1/p)C \log\log(Cm)} B_X,$$

i.e.

$$\delta_X \leq \left(C'm/p\right)^{-(1-1/p)C \log\log(Cm)}.$$

That implies the result. □

Let us conclude this section with a very simple form of Dvoretzky's theorem recast in this language:

Theorem 2.6. *Let $\eta < 1/3$. There is an absolute constant $c > 0$ so that if S is a compact spanning subset of $\mathbb{R}^n$ then there is a projection P of rank at least $c\eta^2 \log n$ such that*

$$d_{\Gamma_\theta PS} \leq \frac{1+\eta}{1-\theta}$$

for every $\sqrt{3\eta} \leq \theta < 1$.

Remark 1. Let $\epsilon \leq 6/7$. Setting $\theta = \sqrt{3\eta} = \epsilon/2$ we observe that there is an absolute constant $c > 0$ so that if S is a compact spanning subset of $\mathbb{R}^n$ then there is a projection P of rank at least $c\epsilon^4 \log n$ such that

$$d_{\Gamma_{\epsilon/2} PS} \leq 1 + \epsilon.$$

Remark 2. The "quotient form" of Dvoretzky's theorem for quasi-normed spaces is essentially known and follows very easily from results in [7] (see e.g. [8] for the details).

Proof. By the sharp form of Dvoretzky's Theorem (Theorem 2.9 in [6]) there is a projection P of rank at least $c\eta^2 \log n$ so that $d_{\Delta(PS)} \le 1+\eta$. Let $Y = P\mathbb{R}^n$ and introduce an inner-product norm $\|\cdot\|$ on Y so that $\mathcal{E} \subset \Delta(PS) \subset (1+\eta)\mathcal{E}$ where $\mathcal{E} = \{y : \ (y,y) \le 1\}$. If $y \in \mathcal{E}$ with $\|y\| = 1$ there exists $u \in PS \cup (-PS)$ with $(y,u) \ge 1$. Since $\|u\| \le 1+\eta$ we obtain $\|y-u\| \le (2\eta+\eta^2)^{1/2} \le \sqrt{3\eta}$. Hence

$$\mathcal{E} \subset PS \cup (-PS) + \sqrt{3\eta}\ \mathcal{E}$$

which implies, for any $\theta \ge \sqrt{3\eta}$,

$$(1-\theta)\mathcal{E} \subset \Gamma_\theta PS \subset (1+\eta)\mathcal{E}.$$

Hence

$$d_{\Gamma_\theta PS} \le \frac{1+\eta}{1-\theta},$$

which proves the theorem. □

3. Approximating the cube

Let n be an integer. By $[n]$ we denote the set $\{1, ..., n\}$. The n-dimensional cube we denote by $B^\infty = B^\infty_n$. D_n denotes the extreme points of the cube, i.e. the set $\{1,-1\}^n$. Given a set $\sigma \subset [n]$ by P_σ we denote the coordinate projection of $\mathbb{R}^n$ onto $\mathbb{R}^\sigma$, and we denote $B^\infty_\sigma := P_\sigma B^\infty_n$, $D_\sigma := P_\sigma D_n$. As above $|A|$ denotes the cardinality of a set A. As usual $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the norm in ℓ_2 and ℓ_∞ correspondingly.

Theorem 3.1. *There are constants $c > 0$ and $0 < C < \infty$ so that for every $\epsilon > 0$, if $S \subset D_n$ with $|S| \ge 2^{n(1-c\epsilon)}$ then there is a subset σ of $[n]$ with $|\sigma| \ge (1-\epsilon)n$ so that*

$$D_\sigma \subset C\epsilon^{-1} P_\sigma(\Delta_N S)$$

for some $N \le C\epsilon^{-2}$.

Proof. We will follow Alesker's argument in [1], which is itself a refinement of Szarek-Talagrand [20]. Alesker shows that for a suitable choice of c, if $\epsilon = 2^{-s}$ then one can find an increasing sequence of subsets $(\sigma_k)_{k=0}^s$ so that $P_{\sigma_0}(S) = D_{\sigma_0}$, $|\sigma_s| \ge (1-2\epsilon)n$ and if $\tau_k = \sigma_k \setminus \sigma_{k-1}$ for $k = 1, 2, \ldots, s$ then there exists $\alpha \in D_n$ so that

$$P_{\tau_k}(S \cap P^{-1}_{\sigma_{k-1}}(P_{\sigma_{k-1}}\alpha)) = D_{\tau_k}.$$

It follows that if $a \in D_{\tau_k}$ there exists $x \in \Delta_2 S$ with $P_{\sigma_{k-1}}(x) = 0$ and $P_{\tau_k}(x) = a$.

We now argue by induction that $D_{\sigma_k} \subset a_k P_{\sigma_k} \Delta_{b_k} S$ where $a_k = 2^{k+1} - 1$ and $b_k = 2^k a_k = 2 \cdot 4^k - 2^k$. This clearly holds if $k = 0$. Assume it is true for $k = j - 1$, where $1 \le j \le s$. Then if $a \in D_{\sigma_j}$ we can observe that there exists $x_1 \in a_{j-1} \Delta_{b_{j-1}} S$ with $P_{\sigma_{j-1}} x_1 = P_{\sigma_{j-1}} a$. Clearly,

$$P_{\tau_j} x_1 \in a_{j-1} \Delta_{b_{j-1}} D_{\tau_j}.$$

Hence there exists $x_2 \in a_{j-1} \Delta_{2b_{j-1}} S$ with $P_{\sigma_{j-1}} x_2 = 0$ and $P_{\tau_j} x_2 = -P_{\tau_j} x_1$. Finally pick $x_3 \in \Delta_2 S$ so that $P_{\sigma_{j-1}}(x_3) = 0$ and $P_{\tau_j}(x_3) = P_{\tau_j} a$. Then $P_{\sigma_j}(x_1 + x_2 + x_3) = a$ and

$$\begin{aligned} x_1 + x_2 + x_3 &\in a_{j-1} \Delta_{b_{j-1}} S + a_{j-1} \Delta_{2b_{j-1}} S + \Delta_2 S \\ &\subset \frac{a_{j-1}}{2b_{j-1}} \left(4b_{j-1} + 2^j\right) \Delta_{4b_{j-1} + 2^j} S = a_j \Delta_{b_j} S. \end{aligned}$$

This establishes the induction.

We finally conclude that $D_{\sigma_s} \subset 2(2^{s+1} - 1) P_{\sigma_s} \Delta_{2 \cdot 4^s} S$ and this gives the result, as the case of general ϵ follows easily. $\square$

Remark. Slightly changing the proof one can show that $D_\sigma \subset C\epsilon^{-\alpha} P_\sigma(\Delta_N S)$ for $N \le C\epsilon^{-\alpha}$, where $\alpha = \log_2 3$.

Lemma 3.2. *There exist absolute constants $c, C > 0$ with the following property. Suppose $0 < \epsilon < 1$ and $0 < k < n$ are natural numbers with $k/n \ge 1 - c\epsilon(1 - \ln \epsilon)^{-1}$. Let S be a subset of $\mathbb{R}^n$ so that if $a \in D_n$ there exists $x \in S$ with $|\{i : x_i = a_i\}| \ge k$. Then there is a subset σ of $[n]$ with $|\sigma| \ge (1 - \epsilon)n$ and $D_\sigma \subset C\epsilon^{-1} \Delta_N P_\sigma S$ for some $N \le C\epsilon^{-2}$.*

Proof. Suppose $0 < k < n$ and $1 - k/n = t\epsilon(1 - \ln \epsilon)^{-1}$. We shall show that if t is small enough we obtain the conclusion of the lemma. First pick a map $a \to \sigma(a)$ from $D_n \to 2^{[n]}$ so that for each a, $|\sigma(a)| = k$ and there exists $x \in S$ with $x_i = a_i$ for $i \in \sigma(a)$. Then, by a simple counting argument we have the existence of $\tau \in 2^{[n]}$ so that $|\tau| = k$ and if

$$T = \{\alpha \in D_\tau : \exists a \in D_n,\ \sigma(a) = \tau,\ P_\tau a = \alpha\}$$

then

$$|T| \ge \frac{2^n}{2^{n-k}\binom{n}{k}}.$$

We can estimate

$$\binom{n}{k} \le \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} \le \left(\frac{ne}{n-k}\right)^{n-k}.$$

Hence for $t \le 1/2$ we have

$$\log_2 \binom{n}{k} \le \frac{nt\epsilon}{\ln 2\,(1-\ln\epsilon)} \ln\left(\frac{e\ln(e/\epsilon)}{t\epsilon}\right) \le 3kt\epsilon\,(2-\ln t)\,.$$

It follows that

$$|T| \ge 2^{k(1-C_t\epsilon)},$$

where $C_t = 3t\,(2-\ln t)$. Choosing t such that $C_t \le c/2$, where c is the constant from Theorem 3.1, and applying this theorem, we obtain the existence of $\sigma \subset \tau$, $|\sigma| \ge (1-\epsilon/2)k \ge (1-\epsilon)n$, with desired property. □

Theorem 3.3. *There are absolute constants $c, C > 0$ such that if $\epsilon > 0$ and S is a subset of $\mathbb{R}^n$ with $B^\infty \subset \Delta S \subset dB^\infty$ then there is a subset σ of $[n]$ with $|\sigma| \ge n(1-\epsilon)$ such that*

$$B^\infty_\sigma \subset (C/\epsilon)\,\Gamma_\theta P_\sigma S$$

for $\theta = 1 - cd^{-2}\epsilon^5(1-\ln\epsilon)^{-1}$.

Proof. Let $\delta = c_1\epsilon$ and m be the smallest integer greater than $c_2 d^2\epsilon^{-3}(1-\ln\epsilon)$, where c_1, c_2 will be chosen later.

Suppose first that $a \in D_n$. Then we can find $N \in \mathbb{N}$, $N \ge m$, and $x_1, \dots, x_N \in S \cup (-S)$ so that

$$\left\| a - \frac{1}{N}(x_1 + \cdots + x_N) \right\|_2^2 \le \frac{nd^2}{m}.$$

Let Ω be the space of all m-subsets of $[N]$ and let μ be normalized counting (probability) measure on Ω. If $(\xi_i)_{i=1}^N$ denote the indicator functions $\xi(\omega) = 1$ if $i \in \omega$ and 0 otherwise then

$$\mathbf{E}(\xi_i) = \mathbf{E}(\xi_i^2) = \frac{m}{N},\ \mathbf{E}(\xi_i\xi_j) = \frac{m(m-1)}{N(N-1)}$$

if $i \ne j$. Thus

$$\mathbf{E}(\xi_i - \mathbf{E}(\xi_i))^2 = \frac{m}{N} - \frac{m^2}{N^2}$$

and

$$\mathbf{E}((\xi_i - \mathbf{E}(\xi_i))(\xi_j - \mathbf{E}(\xi_j)) = \frac{m(m-1)}{N(N-1)} - \frac{m^2}{N^2}$$

if $i \ne j$.

Let $y = \frac{1}{N}(x_1 + \cdots + x_N)$ so that $y = \mathbf{E}(\frac{1}{m}\sum_{i=1}^N \xi_i x_i)$. Then working in the ℓ_2-norm we have

$$\mathbf{E}\left(\left\|\frac{1}{m}\sum_{i=1}^N \xi_i x_i - y\right\|_2^2\right) = \frac{N-m}{mN(N-1)}\sum_{i=1}^N \|x_i\|_2^2 - \frac{N-m}{mN^2(N-1)}\left\|\sum_{i=1}^N x_i\right\|_2^2.$$

Hence

$$\mathbf{E}\left(\left\|\frac{1}{m}\sum_{i=1}^{N}\xi_i x_i - y\right\|_2^2\right) \le \frac{nd^2}{m}.$$

Since $\|y-a\|_2^2 \le \frac{nd^2}{m}$ we have

$$\mathbf{E}\left(\left\|\frac{1}{m}\sum_{i=1}^{N}\xi_i x_i - a\right\|_2^2\right) \le 4\frac{nd^2}{m}.$$

We now suppose that for each $\omega \in \Omega$ we have $|\{j : \ |\frac{1}{m}\sum_{i=1}^N \xi_i x_i(j) - a(j)| > \delta\}| > 4d^2n/(m\delta^2)$. Then we get an immediate contradiction. We conclude that for each $a \in D_n$ there exists $x_a \in \Delta_m S$ such that $|x_a(j) - a(j)| \le \delta$ for at least $n(1 - 2c_1^{-2}c_2^{-1}\epsilon(1-\log\epsilon)^{-1})$ choices of j. Let $y_a(j) = a(j)$ if $|x_a(j) - a(j)| \le \delta$ and $y_a(j) = x_a(j)$ otherwise so that $\|y_a - x_a\|_\infty \le \delta$.

Now suppose c_2 is chosen as a function of c_1 so that we can apply Lemma 3.2 to obtain the existence of a set $\sigma \subset [n]$ with $|\sigma| \ge n(1-\epsilon)$ and so that

$$D_\sigma \subset C\epsilon^{-1}P_\sigma \Delta_N\{y_a : a \in D_n\}$$

where C is an absolute constant, and $N \le C\epsilon^{-2}$. Then

$$D_\sigma \subset C\epsilon^{-1}P_\sigma\Delta_{Nm}S + C\epsilon^{-1}\delta B_\sigma^\infty.$$

Recall that $C\epsilon^{-1}\delta = Cc_1$ so that if we choose c_1 such that $Cc_1 = \frac{1}{4}$ we have

$$D_\sigma \subset K + \frac{1}{4}B_\sigma^\infty$$

where $K := C\epsilon^{-1}P_\sigma\Delta_{Nm}S$. Now suppose $x \in B_\sigma^\infty$. Let $a_1, a_2 \in D_\sigma$ be defined by $a_1(j) = 1$ if $x(j) \ge \frac{1}{2}$ and $a_1(j) = -1$ otherwise, while $a_2(j) = 1$ if $x(j) \ge -\frac{1}{2}$ and $a_2(j) = -1$ otherwise. Then

$$\left\|x - \frac{1}{2}(a_1+a_2)\right\|_\infty \le \frac{1}{2}.$$

Thus

$$B_\sigma^\infty \subset \Delta_2 K + \frac{3}{4}B_\sigma^\infty = C\epsilon^{-1}P_\sigma\Delta_{2Nm}S + \frac{3}{4}B_\sigma^\infty.$$

This implies for $\theta = \frac{3}{4}$,

$$B_\sigma^\infty \subset 4C\epsilon^{-1}\Gamma_\theta P_\sigma \Delta_{2Nm}S$$

Letting $\varphi = \theta^{1/(2Nm)}$ and applying Lemma 2.2 we obtain

$$\Gamma_\theta\Delta_{2Nm}S \subset \frac{6}{5}\Gamma_\varphi S.$$

Note that $(\frac{3}{4})^{1/(2Nm)} \sim 1-(2Nm)^{-1}\ln(4/3) \leq 1-cd^{-2}\epsilon^5(1-\ln\epsilon)^{-1}$ for some $c>0$ so that the result follows. □

Theorem 3.4. *There is an absolute $C>0$ such that if $\epsilon>0$ and X is a p-normed quasi-Banach space with $\dim X = n$ and $d(\hat{X},\ell_\infty^n) \leq d$ then X has a quotient Y with $\dim Y \geq n(1-\epsilon)$ and*

$$d(Y,\ell_\infty^{\dim Y}) \leq Cp^{-\frac{1}{p}}\epsilon^{4-\frac{5}{p}}(1-\ln\epsilon)^{\frac{1}{p}-1}d^{\frac{2}{p}-1}.$$

Remark. In [11] examples are constructed of finite-dimensional p-normed spaces X_n (with $0<p<1$ fixed) so that $d(\hat{X}_n,\ell_\infty^{\dim X_n})$ is uniformly bounded but $\lim_{n\to\infty}\delta_{X_n}=\infty$.

Proof. We can assume $B^\infty \subset B_{\hat{X}} \subset dB^\infty$. Then by Theorem 3.3 we can find σ with $|\sigma| \geq n(1-\epsilon)$ so that

$$c\epsilon B_\sigma^\infty \subset \Gamma_\theta P_\sigma B_X$$

where $\theta = 1-cd^{-2}\epsilon^5(1-\ln\epsilon)^{-1}$. Let Y be the space of dimension $|\sigma|$ with unit ball $B_Y = P_\sigma B_X$. Since B_Y is p-convex we have (Lemma 2.1)

$$\Gamma_\theta B_Y \subset p^{-\frac{1}{p}}(cd^{-2}\epsilon^5(1-\log\epsilon)^{-1})^{1-\frac{1}{p}}B_Y.$$

Finally observe that for a suitable $c>0$:

$$cp^{\frac{1}{p}}d^{2-\frac{2}{p}}\epsilon^{\frac{5}{p}-4}(1-\log\epsilon)^{1-\frac{1}{p}}B_\sigma^\infty \subset B_Y \subset dB_\sigma^\infty.$$

The result then follows. □

4. Cubic quotients

We start this section with the following lemma, which is in fact a corollary of Theorem 3.3.

Lemma 4.1. *Let S be a compact spanning of $\mathbb{R}^n$ and X be the Banach space with unit ball $B_X = \Delta S$. Let m be the largest integer such that X has a subspace Y of dimension m with $d(Y,\ell_1^m) \leq 2$. Then for every integer k satisfying $2^{2k-1} \leq m$ there exists a rank k projection π, so that for some cube Q one has $Q \subset \Gamma_b \pi S \subset CQ$, where $0<b<1$ is an absolute constant.*

Proof. Let Y be a subspace of X of dimension m so that $d(Y,\ell_1^m) \leq 2$. Then if $2^{2k-1} \leq m$ there is a linear operator $T: Y \to \ell_\infty^{2k}$ with $\|T\| \leq 1$ and $T(B_Y) \supset \frac{1}{2}B_{2k}^\infty$. T can then be extended to a norm-one operator on X and so X has a quotient Z of dimension $2k$ so that $d(Z,\ell_\infty^{2k}) \leq 2$. It follows immediately from Theorem 3.3 with $\epsilon = \frac{1}{2}$ that there is a further quotient W of Z with $\dim W \geq k$

and for some cube Q_0 in W, and fixed constants $0 < b < 1$ and $1 < C < \infty$, we have $Q_0 \subset \Gamma_b \pi_W S \subset CQ_0$ where π_W is the quotient map onto W. $\square$

Theorem 4.2. *There is an absolute constant $c > 0$ so that if X is a finite-dimensional p-normed space, then X has a quotient E with $d(E, \ell_\infty^{\dim E}) \leq (cp)^{-1/p}$ and $\dim E \geq c \ln A/(\ln\ln A)$, where $A = (p^{1/p}\delta_X/4)^{p/(1-p)}$ (assuming that δ_X is large enough).*

Remark. Take $X = \ell_p^n$ so that $\delta_X = n^{-1+1/p}$. Then if X has a quotient E of dimension k with $d(E, \ell_\infty^k) \leq C_p$ then $\hat{X} = \ell_1^n$ also has such a quotient which implies $k \leq cC_p \ln n = cC_p \ln\left(\delta_X^{p/(1-p)}\right)$. We conjecture that this estimate is optimal up to an absolute constant, i.e. that every p-normed space has a cubical quotient of such dimension. As one can see from the proof below we could obtain such an estimate (up to constant depending on p only) if we were able to prove the inclusion with $\theta = 1 - c(m\ln m)^{-1}$ in Proposition 2.4.

Proof. Let $S = B_X$ and m be the largest integer such that X has a subspace Y of dimension m with $d(Y, \ell_1^m) \leq 2$.

Assume first $m \leq 2^{2k}$. By Proposition 2.4 (and its proof) we have $\Delta B_X \subset 4\Gamma_\theta B_X$ for $\theta = 2^{-1/N_k}$, where $N_k = (Ck)^{C\ln\ln(Ck)}$. Then, by Lemma 2.1, we obtain

$$\Delta B_X \subset 4p^{-1/p}(2N_k)^{-1+1/p}$$

which implies

$$\delta_X \leq 4p^{-1/p}(2N_k)^{-1+1/p}.$$

Therefore $2N_k \geq A := (p^{1/p}\delta_X/4)^{p/(1-p)}$. Finally we obtain $k \geq C' \ln A/(\ln\ln A)$ (of course we may assume that $A > e^2$).

Suppose now $k \leq C' \ln A/(\ln\ln A)$. By above we have $m \geq 2^{2k}$. So Lemma 4.1 implies the existence of absolute constants b, C_1 and a rank k projection π such that $Q \subset \Gamma_b \pi B_X \subset C_1 Q$ for some cube Q. By Lemma 2.1 we obtain

$$\Gamma_b \pi B_X \subset p^{-1/p}\,(1-b)^{1-1/p}\,\pi B_X$$

so that we have (if $E = X/\pi^{-1}(0)$),

$$d(E, \ell_\infty^k) \leq C_1 p^{-1/p}\,(1-b)^{1-1/p}\,.$$

This implies the theorem. $\square$

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