Abstract. For a given sequence of real numbers \( a_1, \ldots, a_n \) we denote the \( k \)-th smallest one by \( k \)-min\(_{1 \leq i \leq n} a_i \).

We show that there exist two absolute positive constants \( c \) and \( C \) such that for every sequence of positive real numbers \( x_1, \ldots, x_n \) and every \( k \leq n \) one has

\[
c \max_{1 \leq j \leq k} \frac{k + 1 - j}{\sum_{i=j}^{n} 1/x_i} \leq \mathbb{E} \text{k-min}_{1 \leq i \leq n} |x_i g_i| \leq C \ln(k+1) \max_{1 \leq j \leq k} \frac{k + 1 - j}{\sum_{i=j}^{n} 1/x_i},
\]

where \( g_i \in \mathcal{N}(0,1), \ i = 1, \ldots, n, \) are independent Gaussian random variables. Moreover, if \( k = 1 \) then the left hand side estimate does not require independence of the \( g_i \)'s. Similar estimates hold for \( \mathbb{E} \text{k-min}_{1 \leq i \leq n} |x_i g_i|^p \) as well.

Résumé. Pour une suite \( a_1, \ldots, a_n \) des nombres réels, on note le \( k \)-ième plus petit membre par \( k\)-min\(_{1 \leq i \leq n} a_i \). On démontre qu’il existe deux constantes positives \( c \) et \( C \) telles que pour toute suite \( x_1, \ldots, x_n \) des nombres réels et pour tout \( k \leq n \), on ait

\[
c \max_{1 \leq j \leq k} \frac{k + 1 - j}{\sum_{i=j}^{n} 1/x_i} \leq \mathbb{E} \text{k-min}_{1 \leq i \leq n} |x_i g_i| \leq C \ln(k+1) \max_{1 \leq j \leq k} \frac{k + 1 - j}{\sum_{i=j}^{n} 1/x_i}.
\]

Ici \( g_i \in \mathcal{N}(0,1), \ i = 1, \ldots, n, \) sont des variables aléatoires Gaussiennes indépendantes. En plus, si \( k = 1 \), on n’a pas besoin de l’indépendance des \( g_i \)'s pour obtenir l’inégalité du gauche. On démontre également les inégalités correspondantes pour \( \mathbb{E} \text{k-min}_{1 \leq i \leq n} |x_i g_i|^p \).

For a given sequence of real numbers \( (a_i)_{i=1}^{n} \) we denote its non-decreasing rearrangement by \( (k \text{-min}_{1 \leq i \leq n} a_i)_{k=1}^{n} \), thus 1-min\(_{1 \leq i \leq n} a_i = \text{min}_{1 \leq i \leq n} a_i, 2\text{-min}_{1 \leq i \leq n} a_i \) is the next smallest, etc.

Given \( A \subset \mathbb{N} \) we denote its cardinality by \( |A| \). We say that \( (A_j)_{j=1}^{k} \) is a partition of \( \{1, 2, \ldots, n\} \) if \( \emptyset \neq A_j \subset \{1, 2, \ldots, n\}, j \leq k, \cup_{j \leq k} A_j = \{1, 2, \ldots, n\}, \) and \( A_i \cap A_j = \emptyset \) for \( i \neq j \). The canonical

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Euclidean norm and the canonical inner product on $\mathbb{R}^n$ we denote by $|\cdot|_p$ and $\langle \cdot , \cdot \rangle$. By $1/t$ we mean $\infty$ if $t = 0$ and $0$ if $t = \infty$.

In this note we present two theorems. The first one investigates the behavior of the expectation of the minimum of symmetric Gaussian random variables.

**Theorem 1** Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers. Let $g_i \in N(0,1)$, $i \leq n$, be Gaussian random variables. Then

$$\frac{1}{1+p} \left(\frac{\pi}{2}\right)^{p/2} \left(\sum_{i=1}^n |x_i|^{-1}\right)^{-p} \leq \mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p.$$  

Moreover, if the $g_i$'s are independent then

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p \leq \Gamma(1 + p) \left(\frac{\pi}{2}\right)^{p/2} \left(\sum_{i=1}^n |x_i|^{-1}\right)^{-p}.$$

An immediate consequence of this theorem is the following Corollary.

**Corollary 2** Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers. Let $f_i \in N(0,1)$, $i \leq n$, be Gaussian random variables and $g_i \in N(0,1)$, $i \leq n$, be independent Gaussian random variables. Then

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p \leq \Gamma(2 + p) \mathbb{E} \min_{1 \leq i \leq n} |x_i f_i|^p.$$  

**Remark.** This inequality is connected to the Mallat-Zeitouni problem ([2]). In fact, to prove a particular case of Conjecture 1 from [2] it is enough to prove our Corollary for $p = 2$ and with factor 1 instead of $\Gamma(2 + p)$ ([3]). Thus we provide the solution of this case up to constant 6.

Next theorem deals with the moments of $k$-min of independent symmetric Gaussian variables.

**Theorem 3** Let $p > 0$. Let $2 \leq k \leq n$. Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$. Let $g_i \in N(0,1)$, $i \leq n$, be independent Gaussian random variables. Then

$$c_p \max_{1 \leq j \leq k} \frac{k + 1 - j \sum_{i=j}^n 1/x_i}{k \sum_{i=1}^n 1/x_i} \leq \left( \mathbb{E} k \min_{1 \leq i \leq n} |x_i g_i|^p \right)^{1/p} \leq C(p,k) \max_{1 \leq j \leq k} \frac{k + 1 - j \sum_{i=j}^n 1/x_i}{k \sum_{i=1}^n 1/x_i},$$

where $c_p = \frac{1}{2e} \sqrt{\frac{\pi}{2}} \left(1 - \frac{1}{4\sqrt{\pi}}\right)^{1/p}$ and $C(p,k) = 4\sqrt{\pi} \max\{p, \ln(k + 1)\}$.

**Remark.** Theorem 3 shows that we may evaluate sums of the form $\sum_{k \in I} \mathbb{E} k \min_{1 \leq i \leq n} |x_i g_i|^p$, where $I \subset \{1,2,\ldots,n\}$ is any subset of integers. Related inequalities, though in a different context, were developed initially in [1].

Theorems 1 and 3 are consequences of the following Lemmas, which are of independent interest.

**Lemma 4** Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$. Let $g_i \in N(0,1)$, $i \leq n$, be Gaussian random variables. Let $a = \sqrt{2/\pi} \sum_{i=1}^n 1/x_i$. Then for every $t > 0$

$$\mathbb{P}\left\{ \omega \mid \min_{1 \leq i \leq n} |x_i g_i(\omega)| \leq t \right\} \leq at.$$  

Moreover, if the $g_i$'s are independent then for every $t > 0$

$$\mathbb{P}\left\{ \omega \mid \min_{1 \leq i \leq n} |x_i g_i(\omega)| > t \right\} \leq e^{-at}.$$
Lemma 5 Let $1 \leq k \leq n$. Let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$. Let $g_i \in N(0, 1)$, $i \leq n$, be independent Gaussian random variables. Let

$$a = \frac{e}{k} \sqrt{\frac{2}{\pi}} \sum_{i=1}^{n} \frac{1}{x_i}.$$

Then for every $0 < t < 1/a$ one has

$$\mathbb{P}\left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| \leq t \right. \right\} \leq \frac{1}{\sqrt{2\pi k}} \frac{(at)^k}{1-at}. \quad (1)$$

In the rest of this note we provide proofs of Theorems 1 and 3. Proofs of all lemmas will be shown in a forthcoming paper.

Proof of Theorem 1. Let us note that if $x_i = 0$ for some $i$ then the expectation is 0 and the theorem is trivial. Therefore, without loss of generality, we assume that $x_i > 0$ for every $i$.

Denote

$$A = \left(\sqrt{\frac{2}{\pi}} \sum_{k=1}^{n} \frac{1}{x_k}\right)^{-p}.$$

Then, by the first estimate in Lemma 4, we have

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p = \int_{0}^{\infty} \mathbb{P}\left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| > t^{1/p} \right. \right\} dt \geq \int_{0}^{A} \left(1 - t^{1/p} A^{-1/p}\right) dt = \frac{A}{1+p},$$

which proves the first estimate.

Now assume that the $g_i$’s are independent and use the second estimate of Lemma 4. We obtain

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p = \int_{0}^{\infty} \mathbb{P}\left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| > t^{1/p} \right. \right\} dt \leq \int_{0}^{\infty} \exp\left(-t^{1/p} A^{-1/p}\right) dt = A p \Gamma(p),$$

which implies the desired result.

To prove Theorem 3 we need also the following combinatorial lemma.

Lemma 6 Let $1 \leq k \leq n$. Let $(a_i)_{i=1}^{n}$ be a nonincreasing sequence of positive real numbers. Then there exists a partition $(A_l)_{l \leq k}$ of $\{1, 2, \ldots, n\}$ such that

$$\min_{1 \leq l \leq k} \sum_{i \in A_l} a_i \geq \frac{1}{2} \min_{1 \leq l \leq k} \frac{1}{k+1} - j \sum_{i=j}^{n} a_i.$$

Remark. In fact one can show that the $A_l$’s can be taken as intervals, i.e. $A_l = \{i \mid n_l-1 < i \leq n_l\}$, $l \leq k$, for some sequence $0 = n_0 < 1 \leq n_1 < n_2 < \cdots < n_k = n$.

Proof of Theorem 3. First we show the lower estimate. Since for every sequence $(a_i)_{i=1}^{n}$ and every $r < k$ one has

$$k- \min(a_i)_{i=r+1}^{n} \geq (k-r)- \min(a_i)_{i=r+1}^{n},$$

it is enough to show that for every $k$ we have

$$c_p k \left(\sum_{i=1}^{n} \frac{1}{x_i}\right)^{-1} \leq \left(\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p \right)^{1/p}. \quad (2)$$

Let $a$ be as in Lemma 5 and $t = (2a)^{-p}$. Then, by Lemma 5 and since $k \geq 2$, we have

$$\mathbb{P}\left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)|^p \geq t \right. \right\} \geq 1 - \frac{1}{\sqrt{2\pi k}} \frac{(at^{1/p})^k}{1-at^{1/p}} \geq 1 - \frac{1}{4\sqrt{\pi}}.$$
Therefore (2) follows from the standard estimate
\[
\mathbb{E} k^{- \min_{1 \leq i \leq n} |x_i g_i|^p} \geq t^p \mathbb{P} \left\{ \omega \mid k^{- \min_{1 \leq i \leq n} |x_i g_i(\omega)|} \geq t \right\}.
\]

Now we prove the upper bound. Let \((A_j)_{j \leq k}\) be the partition given by Lemma 6 for sequence \(a_i = 1/x_i, i \leq k\). The number \(q, q \geq 1\), will be specified later. It is easy to see that
\[
k^{- \min_{1 \leq i \leq n} |x_i g_i|^p} \leq \max_{j \leq k} \left\{ \min_{i \in A_j} |x_i g_i|^p \right\}.
\]
Therefore, using Theorem 1, we get
\[
\left( \mathbb{E} k^{- \min_{1 \leq i \leq n} |x_i g_i|^p} \right)^{1/p} \leq \left( \mathbb{E} \left( \sum_{j \leq k} \left( \min_{i \in A_j} |x_i g_i|^p \right)^q \right)^{1/q} \right)^{1/p} \leq \left( \mathbb{E} \sum_{j \leq k} \min_{i \in A_j} |x_i g_i|^p q \right)^{1/(pq)}
\]
\[
\leq \sqrt{\frac{\pi}{2}} \left( \Gamma(1 + pq) \sum_{j \leq k} \left( \sum_{i \in A_j} 1/x_i \right)^{-pq} \right)^{1/(pq)} \leq \sqrt{\frac{\pi}{2}} (k \Gamma(1 + pq))^{1/(pq)} \max_{j \leq k} \left( \sum_{i \in A_j} 1/x_i \right)^{-1}.
\]

Applying Lemma 6, we obtain
\[
\left( \mathbb{E} k^{- \min_{1 \leq i \leq n} |x_i g_i|^p} \right)^{1/p} \leq \sqrt{\frac{\pi}{2}} (k \Gamma(1 + pq))^{1/(pq)} \max_{1 \leq j \leq k} \frac{k + 1 - j}{\sum_{i=1}^{n} 1/x_i}.
\]

To complete the proof we choose \(q = \frac{\ln(k+1)}{p}\) if \(p \leq \ln(k+1)\), \(q = 1\) otherwise, and apply Stirling’s formula. \(\square\)

**Remark.** Finally we would like to note that our results can be extended to the case of general distributions satisfying certain conditions. Namely, fix \(\alpha > 0\), \(\beta > 0\) and say that a random variable \(\xi\) satisfies an \((\alpha, \beta)\)-condition if for every \(t > 0\) one has
\[
\mathbb{P} (|\xi| \leq t) \leq \alpha t \quad \text{and} \quad \mathbb{P} (|\xi| > t) \leq e^{-\beta t}.
\]
Then Theorems 1, 3 and Lemmas 4, 5 hold for \(g_i\)’s satisfying an \((\alpha, \beta)\)-condition (even not identically distributed), with constants depending on \(\alpha, \beta\). More precisely, in the estimates of Theorem 1, \(\pi/2)^{p/2}\) should be replaced by \(\alpha^{-p}\) and \(\beta^{-p}\) correspondingly; in Theorem 3, \(\sqrt{\pi/2}\) should be replaced by \(1/\alpha\) and, in the upper estimate, \(4\sqrt{\pi}\) by \(4\sqrt{2}/\beta\); in Lemma 5 and in the first estimate of Lemma 4, \(\sqrt{2/\pi}\) should be replaced by \(\alpha\); in the second estimate of Lemma 4, \(\sqrt{2/\pi}\) should be replaced by \(\beta\).

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**References**

