Packing convex bodies by cylinders *

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Abstract

In [BL] in relation to the unsolved Bang's plank problem (1951) we obtained a lower bound for the sum of relevant measures of cylinders covering a given d-dimensional convex body. In this paper we provide the packing counterpart of these estimates. We also extend bounds to the case of r-fold covering and packing and show a packing analog of Falconer's results ([Fa]).

1 Introduction

In the remarkable paper [Ba] Bang has given an elegant proof of the plank conjecture of Tarski showing that if a convex body is covered by finitely many planks in d-dimensional Euclidean space, then the sum of the widths of the planks is at least as large as the minimal width of the body. We refer to [AKP] for historical remarks and references. A celebrated extension of Bang's theorem to d-dimensional normed spaces has been given by Ball in [B2]. In his paper Bang raises also the important related question whether the sum of the base areas of finitely many cylinders covering a 3-dimensional convex body is at least half of the minimum area of a 2-dimensional projection of the body. In the recent paper [BL] the authors have investigated this problem of Bang in d-dimensional Euclidean space. In particular, we proved Bang's conjecture with constant one third instead of one half. From the point of view of discrete geometry it is quite surprising that so far there has not been any packing analogue of the above theorems on coverings by planks and cylinders. In this paper we fill this gap.

2 Notation

We identify a d-dimensional affine space with $\mathbb{R}^d$. By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^d$. The canonical

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Euclidean ball and sphere in $\mathbb{R}^d$ are denoted by $\mathbf{B}_2^d$ and S^{d-1}. The volume of $\mathbf{B}_2^d$ is denoted by ω_d.

By a convex body $\mathbf{K}$ in $\mathbb{R}^d$ we always mean a compact convex set with the non-empty interior, which is denoted by $\mathrm{int}(\mathbf{K})$. The volume of a convex body $\mathbf{K}$ in $\mathbb{R}^d$ is denoted by $\mathrm{vol}(\mathbf{K})$. When we would like to emphasize that we take d-dimensional volume of a body in $\mathbb{R}^d$ we write $\mathrm{vol}_d(\mathbf{K})$.

The *Banach-Mazur distance* between two convex bodies $\mathbf{K}$ and $\mathbf{L}$ in $\mathbb{R}^d$ is defined by

$$d(\mathbf{K}, \mathbf{L}) = \inf \{\lambda > 0 \ \mid \ a \in \mathbf{L},\ b \in \mathbf{K},\ \mathbf{L} - a \subset T\,(\mathbf{K} - b) \subset \lambda\,(\mathbf{L} - a)\},$$

where the infimum is taken over all (invertible) linear operators $T : \mathbb{R}^d \to \mathbb{R}^d$. The Banach-Mazur distance between $\mathbf{K}$ and the Euclidean ball $\mathbf{B}_2^d$ we denote by $d_{\mathbf{K}}$. John's Theorem ([J]) implies that for every convex body $\mathbf{K}$ in $\mathbb{R}^d$, $d_{\mathbf{K}}$ is bounded by d, moreover if $\mathbf{K}$ is 0-symmetric, i.e., symmetric about the origin 0 in $\mathbb{R}^d$, then $d_{\mathbf{K}} \le \sqrt{d}$ (see e.g. [B1]).

Given a (linear) subspace $E \subset \mathbb{R}^d$ we denote the orthogonal projection on E by P_E and the orthogonal complement of E by $E^{\perp}$. We will use the following theorem, proved by Rogers and Shephard ([RS], see also [C] and Lemma 8.8 in [Pi]).

Theorem 2.1 *Let $1 \le k \le d$. Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and E be a k-dimensional subspace of $\mathbb{R}^d$. Then*

$$\max_{x \in \mathbb{R}^d} \ \mathrm{vol}_{d-k}\left(\mathbf{K} \cap \left(x + E^{\perp}\right)\right) \mathrm{vol}_k(P_E \mathbf{K}) \le \binom{d}{k} \mathrm{vol}_d(\mathbf{K}).$$

Remark. Note that the reverse estimate

$$\max_{x \in \mathbb{R}^d} \ \mathrm{vol}_{d-k}\left(\mathbf{K} \cap \left(x + E^{\perp}\right)\right) \mathrm{vol}_k(P_E \mathbf{K}) \ge \mathrm{vol}_d(\mathbf{K})$$

is a simple consequence of the Fubini Theorem and holds for every measurable set $\mathbf{K}$ in $\mathbb{R}^d$.

3 Preliminary results

Given $0 < k < d$ define a k-codimensional cylinder C as a set, which can be presented in the form $C = B + H$, where H is a k-dimensional (linear) subspace of $\mathbb{R}^d$ and B is a measurable set in $E := H^{\perp}$. Given a convex body $\mathbf{K}$ and a k-codimensional cylinder $C = B + H$ denote the crossectional volume of C with respect to $\mathbf{K}$ by

$$\mathrm{crv}_{\mathbf{K}}(C) := \frac{\mathrm{vol}_{d-k}(C \cap E)}{\mathrm{vol}_{d-k}(P_E \mathbf{K})} = \frac{\mathrm{vol}_{d-k}(P_E C)}{\mathrm{vol}_{d-k}(P_E \mathbf{K})} = \frac{\mathrm{vol}_{d-k}(B)}{\mathrm{vol}_{d-k}(P_E \mathbf{K})}.$$

In [BL] (see Remark 2 following Theorem 3.1 there) we proved that if a convex body $\mathbf{K}$ is covered by k-codimensional cylinders $C_1, \ldots, C_N$, then

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{\binom{d}{k}}. \tag{1}$$

The case $k = d-1$ corresponds to the affine plank problem of Bang ([Ba]), because in this case one has the sum of the relative widths of the planks (i.e., $(d-1)$-codimensional cylinders) on the left side of (1). Note that Ball ([B2]) proved that such sum should exceed 1 in the case of centrally symmetric convex body $\mathbf{K}$, while the general case is still open. The estimate (1) implies the lower bound $1/d$. Moreover, if $\mathbf{K}$ is an ellipsoid and $k = 1$ one has (see Theorem 3.1 in [BL])

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq 1. \tag{2}$$

Akopyan, Karasev and Petrov ([AKP]) have recently proved that (2) holds for 2-codimensional cylinders as well. They have also conjectured that (2) holds for k-codimensional cylinders for all $0 < k < d$.

Before passing to packing, we would like to mention that methods developed in [BL] can be used to prove bounds for multiple coverings. The notion of multiple covering (resp., packing) was introduced in a geometric setting independently by Harold Davenport and László Fejes Tóth [Fe]. Recall that that sets $\mathbf{L}_1, \ldots, \mathbf{L}_N$ form an r-fold covering of $\mathbf{K}$ if every point in $\mathbf{K}$ belongs to at least r of $\mathbf{L}_i$'s. Slightly modifying proofs of Theorem 1 and Remark 2 in [BL], we obtain the following theorem.

Theorem 3.1 *Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and $0 < k < d$. Let $C_1, \ldots, C_N$ be k-codimensional cylinders in $\mathbb{R}^d$ which form an r-fold covering of $\mathbf{K}$. Then*

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq \frac{r}{\binom{d}{k}}.$$

Moreover, if $k = 1$ and $\mathbf{K}$ is an ellipsoid then

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq r.$$

The following multiple covering version of Ball's theorem ([B2]) seems to be an open problem. We note that Falconer ([Fa]) seems to be the first to ask for a multiple covering version of Bang's theorem and proved such a result for convex bodies whose minimal width is two times the inradius (including 0-symmetric convex bodies) in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Conjecture 3.2 *Let $1 < r \le N$ be integers. Let $C_1, \ldots, C_N$ be planks (i.e., $(d-1)$-codimensional cylinders) in $\mathbb{R}^d$ which form an r-fold covering of the 0-symmetric convex body* $\mathbf{K}$. *Then the sum of the relative widths of the planks is at least r, i.e., $\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \ge r$.*

Theorem 3.1 proves the lower bound r/d for r-fold coverings of an arbitrary convex body by planks in $\mathbb{R}^d$. One can restate Conjecture 3.2 with the help of compactness as follows. Recall that for $r = 1$ this was already noticed by Ball in [B2].

Conjecture 3.3 *Let $1 < r \le N$ be integers and let $\mathbf{K}$ be a 0-symmetric convex body in $\mathbb{R}^d$. Then for any family $C_i = H_i + w_i\mathbf{K}$, $i \le N$, of planks with $\sum_{i=1}^{N} w_i = r$, there exists a point $x \in \mathbf{K}$ such that $x \notin \mathrm{int}(C_i)$ for all but at most $r-1$ indices i.*

In fact, the above conjectures are easily seen to be equivalent to the following one.

Conjecture 3.4 *Let $1 < r \le N$ be integers and let $\mathbf{K}$ be a 0-symmetric convex body in $\mathbb{R}^d$. Then for any family H_i, $i \le N$, of hyperplanes in $\mathbb{R}^d$ there exists a point $x \in \frac{N}{N+r}\mathbf{K}_0$ such that $\mathrm{int}\left(x + \frac{r}{N+r}\mathbf{K}_0\right) \cap H_i = \emptyset$ for all but at most $r-1$ indices i.*

The equivalence of Conjectures 3.3 and 3.4 in the case $r = 1$ was already noticed by Alexander in [A].

4 Packing by cylinders

In this section we provide estimates for packing by cylinders in terms of the volumetric parameter, $\mathrm{crv}_{\mathbf{K}}(C)$, introduced in [BL]. Our proofs are close to the proofs of corresponding covering results in Section 3 of [BL]. We provide all the details for the sake of completeness. We start with a definition for a packing by cylinders.

Definition. Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and $C_i = B_i + H_i$, $i \le N$, be k-codimensional cylinders with $1 \le k < d$. Denote $\bar{C}_i = C_i \cap \mathbf{K}$, $i \le N$, and $E_i = H_i^{\perp}$. We say that the C_i's form a packing in $\mathbf{K}$ if $B_i \subset P_{E_i}\mathbf{K}$ for every $i \le N$ and the interiors $\mathrm{int}(\bar{C}_i)$ of $\bar{C}_i$'s are pairwise disjoint. More generally, we say that the C_i's form an r-fold packing in $\mathbf{K}$ if $B_i \subset P_{E_i}\mathbf{K}$ for every $i \le N$ and each point of $\mathbf{K}$ belongs to at most r of $\mathrm{int}(\bar{C}_i)$'s. Clearly, a 1-fold packing is just a packing.

First we provide estimates in the case of 1-codimensional cylinders. Recall here that $d_{\mathbf{K}}$ denotes the Banach-Mazur distance to the Euclidean ball and ω_n denotes the volume of $\mathbf{B}_2^n$.

Theorem 4.1 *Let* $\mathbf{K}$ *be an ellipsoid in* $\mathbb{R}^d$*. Let* $C_1, \ldots, C_N$ *be* 1*-codimensional cylinders in* $\mathbb{R}^d$*, which form an* r*-fold packing in* $\mathbf{K}$*. Then*

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \le r. \tag{3}$$

Remark. This theorem can be used to get bounds in the general case as well. Indeed, let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and T be an invertible linear transformation satisfying

$$d_{\mathbf{K}}^{-1} T\mathbf{B}_2^d \subset \mathbf{K} \subset T\mathbf{B}_2^d.$$

Let $C_1, \ldots, C_N$ be 1-codimensional cylinders in $\mathbb{R}^d$ forming an r-fold packing in $T\mathbf{B}_2^d$. Then, using definitions and Theorem 4.1, we observe

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) = \sum_{i=1}^{N} \frac{\mathrm{vol}_{d-1}(B_i)}{\mathrm{vol}_{d-1}(P_{E_i}\mathbf{K})} \le \sum_{i=1}^{N} \frac{\mathrm{vol}_{d-1}(B_i)}{\mathrm{vol}_{d-1}(P_{E_i} d_{\mathbf{K}}^{-1} T\mathbf{B}_2^d)}$$

$$= d_{\mathbf{K}}^{d-1} \sum_{i=1}^{N} \mathrm{crv}_{T\mathbf{B}_2^d}(C_i) \le r d_{\mathbf{K}}^{d-1}.$$

Proof: Every C_i can be presented as $C_i = B_i + \ell_i$, where ℓ_i is a line containing 0 in $\mathbb{R}^d$ and B_i is a body in $E_i := \ell_i^{\perp}$ such that $B_i \subset P_{E_i}\mathbf{K}$.

Since $\mathrm{crv}_{\mathbf{K}}(C) = \mathrm{crv}_{T\mathbf{K}}(TC)$ for every invertible affine map $T : \mathbb{R}^d \to \mathbb{R}^d$, we may assume that $\mathbf{K} = \mathbf{B}_2^d$. Then

$$\mathrm{crv}_{\mathbf{K}}(C_i) = \frac{\mathrm{vol}_{d-1}(B_i)}{\omega_{d-1}}.$$

Consider the following (density) function on $\mathbb{R}^d$

$$p(x) = 1/\sqrt{1 - |x|^2}$$

for $|x| < 1$ and $p(x) = 0$ otherwise. The corresponding measure on $\mathbb{R}^d$ we denote by μ, that is $d\mu(x) = p(x)dx$. Let ℓ be a line containing 0 in $\mathbb{R}^d$ and $E = \ell^{\perp}$. It follows from direct calculations that for every $z \in E$ with $|z| < 1$

$$\int_{\ell+z} p(x)\ dx = \pi.$$

Thus we have

$$\mu(\mathbf{B}_2^d) = \int_{\mathbf{B}_2^d} p(x)\ dx = \int_{\mathbf{B}_2^d \cap E} \int_{\ell+z} p(x)\ dx\ dz = \pi\ \omega_{d-1}$$

and for every $i \le N$

$$\mu(C_i) = \mu\left(C_i \cap \mathbf{B}_2^d\right) = \int_{C_i} p(x)\ dx = \int_{B_i} \int_{\ell_i+z} p(x)\ dx\ dz = \pi\ \mathrm{vol}_{d-1}\left(B_i\right).$$

Since each point of $\mathbf{K}$ belongs to at most r of $\mathrm{int}(\bar{C}_i)$'s, where $\bar{C}_i = C_i \cap \mathbf{B}_2^d$, $i \leq N$, we obtain that

$$r\pi\,\omega_{d-1} = r\mu(\mathbf{B}_2^d) \geq \mu\left(\bigcup_{i=1}^N \bar{C}_i\right) = \sum_{i=1}^N \mu\left(\bar{C}_i\right) = \sum_{i=1}^N \pi\ \mathrm{vol}_{d-1}\left(B_i\right).$$

This implies

$$\sum_{i=1}^N \mathrm{crv}_{\mathbf{B}_2^d}(C_i) = \sum_{i=1}^N \frac{\mathrm{vol}_{d-1}(B_i)}{\omega_{d-1}} \leq r, \tag{4}$$

which completes the proof. □

Now recall the following idea from [AKP]. Consider the density function defined on $\mathbb{R}^d$ as follows: $p(x) = 1$ for $|x| = 1$ and $p(x) = 0$ otherwise. The corresponding measure on $\mathbb{R}^d$ we denote by μ, that is $d\mu(x) = p(x)d\lambda(x)$, where λ is the Lebesgue measure on S^{d-1}. Let H be a plane containing 0 in $\mathbb{R}^d$ and $E = H^\perp$. Then for every $z \in E$ with $|z| < 1$

$$\int_{H+z} p(x)\ dx = 2\pi.$$

Therefore repeating the proof of Theorem 4.1 with respect to the just introduced density function, we obtain the following theorem.

Theorem 4.2 *Let $\mathbf{K}$ be an ellipsoid in $\mathbb{R}^d$. Let $C_1, \ldots, C_N$ be 2-codimensional cylinders in $\mathbb{R}^d$, which form an r-fold packing in $\mathbf{K}$. Then*

$$\sum_{i=1}^N \mathrm{crv}_{\mathbf{K}}(C_i) \leq r. \tag{5}$$

Remark. As in the case of 1-codimensional cylinders, this theorem can be generalized in the following way. Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and T be an invertible linear transformation T satisfying $d_{\mathbf{K}}^{-1} T\mathbf{B}_2^d \subset \mathbf{K} \subset T\mathbf{B}_2^d$. Let $C_1, \ldots, C_N$ be 2-codimensional cylinders in $\mathbb{R}^d$ forming an r-fold packing in $T\mathbf{B}_2^d$. Then

$$\sum_{i=1}^N \mathrm{crv}_{\mathbf{K}}(C_i) \leq r d_{\mathbf{K}}^{d-2}. \tag{6}$$

The next theorem deals with k-codimensional convex cylinders.

Theorem 4.3 *Let $0 < k < d$ and $\mathbf{K}$ be a convex body in $\mathbb{R}^d$. Let $C_i = B_i + H_i$, $i \leq N$, be k-codimensional cylinders in $\mathbb{R}^d$, which form an r-fold packing in $\mathbf{K}$ and let $\bar{C}_i = C_i \cap \mathbf{K}$. Assume that $\bar{C}_i$'s are convex bodies in $\mathbb{R}^d$. Then*

$$\sum_{i=1}^N \mathrm{crv}_{\mathbf{K}}(C_i) \leq r \binom{d}{k} \max_{i \leq N} \frac{\max_{x\in\mathbb{R}^d} \mathrm{vol}_k(\mathbf{K} \cap (x + H_i))}{\max_{x\in\mathbb{R}^d} \mathrm{vol}_k(\bar{C}_i \cap (x + H_i))}.$$

Proof: As before denote $E_i = H_i^\perp$ and $B_i = P_{E_i} C_i$. As $\bar{C}_i$'s form a packing in $\mathbf{K}$ we have $B_i \subset P_{E_i}\mathbf{K}$ and hence $P_{E_i}\bar{C}_i = B_i$.

Applying Theorem 2.1 and remark following it, we obtain for every $1 \le i \le N$

$$\mathrm{crv}_{\mathbf{K}}(C_i) = \frac{\mathrm{vol}_{d-k}(B_i)}{\mathrm{vol}_{d-k}(P_{E_i}\mathbf{K})} = \frac{\mathrm{vol}_{d-k}(P_{E_i}\bar{C}_i)}{\mathrm{vol}_{d-k}(P_{E_i}\mathbf{K})}$$

$$\le \binom{d}{k} \frac{\mathrm{vol}_d(\bar{C}_i)}{\max_{x\in\mathbb{R}^d}\ \mathrm{vol}_k\left(\bar{C}_i \cap (x+H_i)\right)} \ \frac{\max_{x\in\mathbb{R}^d}\ \mathrm{vol}_k\left(\mathbf{K} \cap (x+H_i)\right)}{\mathrm{vol}_d(\mathbf{K})}.$$

Since the C_i's form an r-fold packing in $\mathbf{K}$, we observe that

$$\sum_{i=1}^N \mathrm{vol}_d\left(\bar{C}_i\right) \le r\,\mathrm{vol}_d\left(\mathbf{K}\right),$$

which implies the desired result. □

Finally we show an example showing some restrictions on the upper bound. We need the following simple lemma.

Lemma 4.4 *For every $\delta \in (0, \pi/2)$ and every $n \ge 1$ one has*

$$\frac{\delta\,(\sin\delta)^n}{e(n+1)} \le \int_{\pi/2-\delta}^{\pi/2} (\cos t)^n\,dt \le \delta\,(\sin\delta)^n.$$

Proof: The upper estimate is trivial, as $\cos(\cdot)$ is a decreasing function on $(0, \pi/2)$. For the lower bound note that $\sin(\beta\delta) \ge \beta\sin\delta$ for $\beta \in (0,1)$ and therefore

$$\int_{\pi/2-\delta}^{\pi/2} (\cos t)^n\,dt \ge \int_{\pi/2-\delta}^{\pi/2-\beta\delta} (\cos t)^n\,dt \ge (1-\beta)\delta\,(\sin(\beta\delta))^n \ge (1-\beta)\delta\beta^n(\sin\delta)^n.$$

The choice $\beta = n/(n+1)$ completes the proof. □

The next theorem shows that sum of $\mathrm{crv}_{\mathbf{K}}(C_i)$ cannot be too small in general. The example is based on a packing of cylinders, whose bases are caps in the Euclidean ball. We will use the following notation. Given m-dimensional subspace E in $\mathbb{R}^d$, $\delta \in (0, \pi/2)$ and $x \in S^{d-1} \cap E$ we denote

$$S(x,\delta,E) := \{z \in \mathbf{B}_2^d \cap E \ \mid\ |\langle z, x\rangle| \ge \cos\delta\}.$$

In other words, $S(x,\delta,E)$ is a (solid) cap in the Euclidean ball in E with the center at x and the (geodesic) radius δ. We also denote

$$S(x,\delta) := \{z \in S^{d-1} \ \mid\ |\langle z, x\rangle| \ge \cos\delta\},$$

that is $S(x,\delta)$ is a (spherical) cap in the Euclidean ball in $\mathbb{R}^d$.

Theorem 4.5 *Let $d > 3$, $1 \leq k < d$ and $\delta \in (0, \pi/4)$. There exist k-codimensional cylinders $C_1, \ldots, C_N$ in $\mathbb{R}^d$, which form a packing in $\mathbf{B}_2^d$ and satisfy*

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{B}_2^d}(C_i) = \frac{1}{\omega_{d-k}} \sum_{i=1}^{N} \mathrm{vol}_{d-k}(B_i) \geq \frac{c\sqrt{d}\,(\sin\delta)^{2-k}}{2^{d-2}\,(d-k)^{3/2}},$$

where c is an absolute positive constant.

Remark. The proof of Theorem 4.5 below uses representations of caps $S(x, \delta)$ as k-codimensional cylinders $C(x) = S(x, \delta, E_x) + E_x^{\perp}$. Then $\bar{C}(x) = C(x) \cap \mathbf{B}_2^d = \mathrm{conv}\ S(x, \delta)$ and it is not difficult to see that the maximum over $y \in \mathbb{R}^d$ of $\mathrm{vol}_k(\bar{C}(x) \cap (y + E_x^{\perp}))$ attains at $y = \cos(\delta)x$ and equals

$$\mathrm{vol}_k(y + \sqrt{1 - |y|^2}\,\mathbf{B}_2^d \cap E_x^{\perp})) = (\sin\delta)^k \omega_k.$$

Thus, for such cylinders the upper bound from Theorem 4.3 becomes

$$\binom{d}{k} (\sin\delta)^{-k}.$$

Therefore, in this example, the ratio between the upper and lower bounds is of the order $C(d, k)(\sin\delta)^{-2}$.

Proof: Given $x \in S^{d-1}$ we construct a k-codimensional cylinder $C(x)$ in the following way. Fix a $(d-k)$-dimensional subspace E_x containing x. Let $C(x) = S(x, \delta, E_x) + E_x^{\perp}$. Of course $C(x)$ depends on the choice of E_x, so for every x we fix one such E_x. With such a construction we have

$$\bar{C}(x) = C(x) \cap \mathbf{B}_2^d = S(x, \delta, \mathbb{R}^d) = \mathrm{conv}\ S(x, \delta).$$

Note that using the Fubini theorem and substitution $x = \sin t$, one has

$$\mathrm{vol}_{d-k}(S(x, \delta, E)) = \int_{\cos\delta}^{1} (1 - x^2)^{\frac{d-k-1}{2}} \omega_{d-k-1}\, dx = \omega_{d-k-1} \int_{\pi/2-\delta}^{\pi/2} (\cos t)^{d-k}\, dt$$

and similarly $\omega_d = \omega_{d-1} \int_{-\pi/2}^{\pi/2} (\cos t)^d\, dt$.

Now we construct a packing of caps in S^{d-1} and estimate its cardinality using standard volumetric argument. Let $\{x_i\}_{i \leq N}$ be a maximal (in sense of inclusion) (2δ)-separated set, that is the geodesic distance between x_i and x_j is larger than 2δ whenever $i \neq j$. Then clearly caps $S(x_i, \delta)$ are pairwise disjoint, hence so are $S(x_i, \delta, \mathbb{R}^d)$'s. Therefore, cylinders $C(x_i)$ form a packing of $\mathbf{B}_2^d$. On the other hand, due to the maximality of the set $\{x_i\}_{i \leq N}$, the caps $S(x_i, 2\delta)$ cover S^{d-1}. Therefore

$$1 \leq \sum_{i=1}^{N} \sigma(S(x_i, 2\delta)) = N\sigma(S(x_1, 2\delta)),$$

where σ is the normalized Lebesgue measure of the sphere S^{d-1}. Thus $N \geq (\sigma(S(x_1, 2\delta)))^{-1}$. The measure of a spherical cap can be directly calculated as (see e.g. Chapter 2 of [MS])

$$\sigma(S(x_1, 2\delta)) = \frac{\int_{\pi/2-2\delta}^{\pi/2} (\cos t)^{d-2}\, dt}{\int_{-\pi/2}^{\pi/2} (\cos t)^{d-2}\, dt} = \frac{\omega_{d-3}}{\omega_{d-2}} \int_{\pi/2-2\delta}^{\pi/2} (\cos t)^{d-2}\, dt.$$

Finally we obtain that there are N cylinders $C(x_i)$, which form packing and

$$A := \sum_{i=1}^{N} \mathrm{crv}_{\mathbf{B}_2^d}(C_i) = N \, \frac{\mathrm{vol}_{d-k}(S(x, \delta, E))}{\omega_{d-k}} \geq \frac{\omega_{d-k-1}}{\omega_{d-k}} \, \frac{\omega_{d-2}}{\omega_{d-3}} \, \frac{\int_{\pi/2-\delta}^{\pi/2} (\cos t)^{d-k}\, dt}{\int_{\pi/2-2\delta}^{\pi/2} (\cos t)^{d-2}\, dt}.$$

Using Lemma 4.4 and estimates for the volume of the Euclidean ball, we observe for an absolute positive constant c,

$$A \geq \frac{c\sqrt{d}}{(d-k)^{3/2}} \, \frac{(\sin \delta)^{d-k}}{(\sin(2\delta))^{d-2}} \geq \frac{c\sqrt{d}\,(\sin \delta)^{2-k}}{2^{d-2}\,(d-k)^{3/2}}.$$

□

5 More on packing by cylinders

In this section we estimate the total volume of bases of 1-codimensional cylinders forming a multiple packing in a given convex body.

Theorem 5.1 *Let* $\mathbf{K}$ *be a convex body in* $\mathbb{R}^d$*. For* $i \leq N$ *let* $C_i = B_i + H_i$ *be* 1*-codimensional cylinders in* $\mathbb{R}^d$*, which form an* r*-fold packing in* $\mathbf{K}$*. Then*

$$\sum_{i=1}^{N} \mathrm{vol}_{d-1}(B_i) \leq c_d\, r \max_{\dim L = d-1} \mathrm{vol}_{d-1}(P_L \mathbf{K}), \tag{7}$$

where $c_d = d\,\omega_d/(2\omega_{d-1}) \sim \sqrt{\pi\, d/2}$ *(as* d *grows to infinity).*

Proof: We use Cauchy formula for the surface area of $\mathbf{K}$:

$$s(\mathbf{K}) = \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \mathrm{vol}_{d-1}\left(P_{u^\perp} \mathbf{K}\right) d\lambda(u),$$

where $d\lambda(\cdot)$ is the Lebesgue measure on S^{d-1}.

For $i \leq N$ denote $\bar{C}_i = C_i \cap \mathbf{K}$. As C_i's form an r-fold packing in $\mathbf{K}$ we have that $(\bar{C}_i \cap \mathrm{bd}\mathbf{K})$'s form an r-fold packing on the boundary $\mathrm{bd}\mathbf{K}$ of $\mathbf{K}$ and therefore

$$\sum_{i=1}^{N} s(\bar{C}_i \cap \mathrm{bd}\mathbf{K}) \leq r \cdot s(K).$$

Using that $\mathrm{vol}_{d-1}(B_i) \leq \frac{1}{2}\, s(\bar{C}_i \cap \mathrm{bd}\mathbf{K})$ and that $\lambda(S^{d-1}) = d\omega_d$, we obtain

$$2\sum_{i=1}^{N}\mathrm{vol}_{d-1}(B_i) \le r\cdot s(\mathbf{K}) = \frac{r}{\omega_{d-1}}\int_{S^{d-1}}\mathrm{vol}_{d-1}\left(P_{u^\perp}\mathbf{K}\right)d\lambda(u)$$

$$\le r\frac{d\,\omega_d}{\omega_{d-1}}\max_{\dim L=d-1}\mathrm{vol}_{d-1}(P_L\mathbf{K}),$$

finishing the proof of Theorem 5.1. □

Remark. It would be interesting to find the best possible value of c_d (as a function of d) in Theorem 5.1. Note that Theorem 4.1 implies that when $\mathbf{K}$ is an ellipsoid one can take $c_d = 1$. This leads to another natural problem: Provide a characterization of convex bodies in $\mathbb{R}^d$ that satisfy Theorem 5.1 with c_d bounded by an absolute constant.

6 Packing counterpart of Falconer's bounds

In this section we provide a packing counterpart of Falconer's bounds. Recall that Falconer ([Fa]) gave an elegant analytic proof of the following multiple covering version of Bang's theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$. Let $\mathbf{K}$ be a convex body in $\mathbb{R}^2$ or $\mathbb{R}^3$ (i.e., a convex domain) whose minimal width $\mathrm{w}(\mathbf{K})$ is equal to the diameter of its incircle (note that any 0-symmetric convex domain has this property). If finitely many planks form an r-fold covering of $\mathbf{K}$, then the sum of the widths of the planks is at least $r\,\mathrm{w}(\mathbf{K})$. Here we provide the packing counterpart of Falconer's estimate in $\mathbb{R}^2$. Following Hadwiger ([H]) we say, that a finite family of closed circular disks form a separable arrangement in $\mathbb{R}^2$ if there exists a line in $\mathbb{R}^2$ that is disjoint from all the disks and divides the plane into two open half-planes each containing at least one disk. In the contrary case we shall call the family a non-separable arrangement (in short, an NSA-family) in $\mathbb{R}^2$. In other words, a finite family of closed circular disks form a non-separable arrangement (i.e., an NSA-family) in $\mathbb{R}^2$ if no line of $\mathbb{R}^2$ divides the disks into two non-empty sets without touching or intersecting at least one disk. We call the convex hull (resp., the sum of the diameters) of an NSA-family of disks an NSA-domain (NSA-diameter). The following theorem improves Theorem 5.1 (with $c_2 = 1$) for NSA-domains $\mathbf{K}$ whose NSA-diameter $\mathrm{diam}_{NSA}(\mathbf{K})$ satisfies $\mathrm{diam}_{NSA}(\mathbf{K}) = 2R_{\mathbf{K}} = \mathrm{diam}(\mathbf{K})$, where $R_{\mathbf{K}}$ denotes the radius of the smallest circular disk containing $\mathbf{K}$ (also called the circumradius of $\mathbf{K}$) and $\mathrm{diam}(\mathbf{K})$ denotes the Euclidean diameter of $\mathbf{K}$.

Theorem 6.1 *Let $\mathbf{K}$ be an arbitrary NSA-domain in $\mathbb{R}^2$ with NSA-diameter $\mathrm{diam}_{NSA}(\mathbf{K})$. If finitely many planks form an r-fold packing in $\mathbf{K}$, then the sum of the widths of the planks is at most $r\mathrm{diam}_{NSA}(\mathbf{K})$. Here, $2R_{\mathbf{K}} \le \mathrm{diam}_{NSA}(\mathbf{K})$ with equality if and only if $2R_{\mathbf{K}} = \mathrm{diam}_{NSA}(\mathbf{K}) = \mathrm{diam}(\mathbf{K})$.*

Our proof is a packing analogue of Falconer's analytic method introduced for coverings by planks in [Fa]. The core part of the discussions that follow is in $\mathbb{R}^d$ and might be of independent interest. Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and let

$$\mathcal{L}^+(\mathbf{K}) = \{f : \mathbf{K} \to \mathbb{R}^+ \mid f \geq 0 \text{ and Lebesgue integrable over } \mathbf{K}\}.$$

Moreover, let $H(s,u) = \{x \in \mathbb{R}^d \mid \langle x, u\rangle = s\}$ denote the hyperplane in $\mathbb{R}^d$ with normal vector $u \in S^{d-1}$ lying at distance $s \geq 0$ from the origin 0. Furthermore, let the sectional integral of f over $H(s,u) \cap \mathrm{int}(\mathbf{K})$ be denoted by

$$F(f,s,u) = \int_{H(s,u)\cap\mathrm{int}(\mathbf{K})} f(x)\ d_{H(s,u)}x$$

for any $H(s,u)$ with $H(s,u) \cap \mathrm{int}(\mathbf{K}) \neq \emptyset$ and with respect to the corresponding $(d-1)$-dimensional Lebesgue measure over $H(s,u)$. Moreover, for $\Delta > 0$ let

$$\mathcal{L}^+_\Delta(\mathbf{K}) = \{f \in \mathcal{L}^+(\mathbf{K}) \mid F(f,s,u) \geq \Delta \text{ for all } H(s,u) \text{ with } H(s,u)\cap\mathrm{int}(\mathbf{K}) \neq \emptyset\}.$$

Finally, let

$$m(\mathcal{L}^+_\Delta(\mathbf{K})) = \inf\left\{\int_{\mathbf{K}} f(x)dx \mid f \in \mathcal{L}^+_\Delta(\mathbf{K})\right\}.$$

If g is a Lebesgue integrable function on $\mathbb{R}$ then $g(\langle x, u\rangle)$, $x \in \mathbb{R}^d$, is called a ridge function in the direction $u \in S^{d-1}$.

Lemma 6.2 *Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and let $u_i \in S^{d-1}$, $1 \leq i \leq N$. Let $g_i(\langle x, u_i\rangle)$, $1 \leq i \leq N$, be ridge functions such that the support of g_i is contained in $[a_i, b_i]$, where*

$$a_i = \min\{\langle x, u_i\rangle \ \mid x \in \mathbf{K}\} \quad \textit{and} \quad b_i = \max\{\langle x, u_i\rangle \ \mid x \in \mathbf{K}\}.$$

Assume that for every $x \in \mathbf{K}$,

$$\sum_{i=1}^{N} g_i(\langle x, u_i\rangle) \leq 1.$$

Then

$$\sum_{i=1}^{N} \int_{-\infty}^{+\infty} g_i(t)dt \leq m(\mathcal{L}^+_1(\mathbf{K})).$$

Proof: For every $f \in \mathcal{L}^+_1(\mathbf{K})$ one has

$$\sum_{i=1}^{N} \int_{-\infty}^{+\infty} g_i(t)dt \leq \sum_{i=1}^{N} \int_{-\infty}^{+\infty} g_i(t)F(f,t,u_i)dt$$

$$= \int_{\mathbf{K}} f(x) \sum_{i=1}^{N} g_i(\langle x, u_i\rangle)dx \leq \int_{\mathbf{K}} f(x)dx,$$

which implies the desired result. □

Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and let $\mathbf{B}_2^d(\mathbf{K})$ denote the circumscribed ball of $\mathbf{K}$, i.e. the smallest Euclidean ball containing $\mathbf{K}$ with radius $R_{\mathbf{K}}$, called the circumradius of $\mathbf{K}$. Recall that the support function of $\mathbf{K}$ is defined by

$$h_{\mathbf{K}}(x) = \sup\{\langle x, k\rangle \ \mid k \in \mathbf{K}\}$$

for $x \in \mathbb{R}^d$.

Lemma 6.3 *If $\mathbf{K}$ is a convex body with circumradius $R_{\mathbf{K}}$ in $\mathbb{R}^d$, then*

$$m(\mathcal{L}_\Delta^+(\mathbf{K})) \geq 2\Delta R_{\mathbf{K}}.$$

Proof: Let $f \in \mathcal{L}_\Delta^+(\mathbf{K})$. As $\int_{\mathbf{K}} f(x)dx > 0$, $\mathbf{K}$ weighted by f has a centroid, i.e., a point c such that

$$\int_{\mathbf{K}} f(x)(x-c)dx = 0,$$

which lies inside $\mathbf{K}$, and which we will take to be the origin 0. Let $h_{\mathbf{K}}$ be the support function of $\mathbf{K}$. The above definition of $\mathbf{B}_2^d(\mathbf{K})$ (resp., $R_{\mathbf{K}}$) implies in a straightforward way that for every $x \in S^{d-1}$, $h_{\mathbf{K}}(x) \leq R_{\mathbf{K}}$ and that there exists $u \in S^{d-1}$ with

$$h_{\mathbf{K}}(u) = R_{\mathbf{K}}. \tag{8}$$

Taking moments perpendicular to u yield

$$\int_{-h_{\mathbf{K}}(-u)}^{h_{\mathbf{K}}(u)} tF(f,t,u)dt = 0. \tag{9}$$

Using $F(f,t,u) \geq \Delta$ and (8) we observe

$$\int_0^{h_{\mathbf{K}}(u)} tF(f,t,u)dt \geq \frac{1}{2}\Delta R_{\mathbf{K}}^2. \tag{10}$$

Then (9) and (10) yield that

$$\int_0^{h_{\mathbf{K}}(-u)} tF(f,t,u)dt \geq \frac{1}{2}\Delta R_{\mathbf{K}}^2. \tag{11}$$

One can check that

$$\inf\left\{\int_0^A F(t)dt \mid A > 0,\, F(t) \geq \Delta \text{ and } \int_0^{+\infty} tF(t)dt \geq M\right\} = (2M\Delta)^{\frac{1}{2}} \tag{12}$$

with the infimum being attained if and only if $F(t) = \Delta$ for (almost) all $t \leq A$ and $A = (\frac{2M}{\Delta})^{\frac{1}{2}}$. Thus (10), (11), and (12) yield

$$\int_{\mathbf{K}} f(x)dx = \int_{-h_{\mathbf{K}}(-u)}^{h_{\mathbf{K}}(u)} F(f,t,u)dt \geq 2\left(2\left(\frac{1}{2}\Delta R_{\mathbf{K}}^2\right)\Delta\right)^{\frac{1}{2}} = 2\Delta R_{\mathbf{K}},$$

finishing the proof of Lemma 6.3. $\square$

Proof of Theorem 6.1: Let $\mathbf{K}$ be an NSA-domain in $\mathbb{R}^2$ with NSA-diameter $\mathrm{diam}_{NSA}(\mathbf{K})$ and let $C_1, \ldots, C_N$ be planks that form an r-fold packing in $\mathbf{K}$. For every $1 \le i \le N$, choose $u_i \in S^{d-1}$ which is orthogonal C_i and let

$$a_i = \min\{\langle x, u_i\rangle \ \mid x \in C_i\} \quad \text{and} \quad b_i = \max\{\langle x, u_i\rangle \ \mid x \in C_i\}.$$

Without loss of generality we assume $a_i \le b_i$. Clearly, the Euclidean width $w(C_i)$ of the plank C_i satisfies $w(C_i) = b_i - a_i$. Consider the ridge functions

$$g_i(\langle x, u_i\rangle) = \frac{1}{r}\,\chi_{[a_i,b_i]}(\langle x, u_i\rangle),$$

where $\chi_{[a_i,b_i]}$ is the characteristic function of the segment $[a_i, b_i]$. On the one hand, Lemma 6.2 applied to g_i's implies that

$$\sum_{i=1}^{N} w(C_i) \le rm(\mathcal{L}_1^+(\mathbf{K})). \tag{13}$$

On the other hand, recall the following well-known fact (Theorem 1.2 in [Fa]): $m(\mathcal{L}_1^+(R\mathbf{B}_2^d)) = 2R$ and this value is attained uniquely by the function $f(x) = \frac{1}{\pi R}(R^2 - |x|^2)^{-\frac{1}{2}}$ if $|x| < R$ and $f(x) = 0$ if $|x| \ge R$. By taking the sum of the analogue functions over the generating circular disks of the NSA-domain $\mathbf{K}$ we get that

$$m(\mathcal{L}_1^+(\mathbf{K})) \le \mathrm{diam}_{NSA}(\mathbf{K}). \tag{14}$$

This and (13) yield $\sum_{i=1}^{N} w(C_i) \le r\mathrm{diam}_{NSA}(\mathbf{K})$. Moreover, Lemma 6.3 and (14) imply that

$$2R_{\mathbf{K}} \le \mathrm{diam}_{NSA}(\mathbf{K}). \tag{15}$$

For a completely different proof of (15) we refer the interested reader to Goodman and Goodman ([GG]). As the case of equality is rather straightforward to show, this completes the proof of Theorem 6.1. $\square$

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