A short proof of Paouris’ inequality

Radosław Adamczak ∗       Rafał Latała †
Alexander E. Litvak ‡       Krzysztof Oleszkiewicz §
Alain Pajor ¶       Nicole Tomczak-Jaegermann ∥

Abstract

We give a short proof of a result of G. Paouris on the tail behaviour of the Euclidean norm \(|X|\) of an isotropic log-concave random vector \(X \in \mathbb{R}^n\), stating that for every \(t \geq 1\),

\[ \mathbb{P}(|X| \geq ct\sqrt{n}) \leq \exp(-t\sqrt{n}). \]

More precisely we show that for any log-concave random vector \(X\) and any \(p \geq 1\),

\[ (\mathbb{E}|X|^p)^{1/p} \sim \mathbb{E}|X| + \sup_{z \in S^{n-1}} (\mathbb{E}(|\langle z, X \rangle|^p)^{1/p}). \]

AMS Classification: 46B06, 46B09 (Primary), 52A23 (Secondary)

Key Words and Phrases: log-concave random vectors, deviation inequalities

∗Research partially supported by MNiSW Grant no. N N201 397437.
†Research partially supported by MNiSW Grant no. N N201 397437.
‡Research partially supported by the E.W.R. Steacie Memorial Fellowship.
§Research partially supported by MNiSW Grant no. N N201 397437.
¶Research partially supported by the ANR project ANR-08-BLAN-0311-01.
∥This author holds the Canada Research Chair in Geometric Analysis.
1 Introduction

Let $X$ be a random vector in the Euclidean space $\mathbb{R}^n$ equipped with its Euclidean norm $| \cdot |$ and its scalar product $\langle \cdot, \cdot \rangle$. Assume that $X$ has a log-concave distribution (a typical example of such a distribution is a random vector uniformly distributed on a convex body). Assume further that it is centered and its covariance matrix is the identity; such a random vector will be called isotropic.

A famous and important result of G. Paouris ([14], Theorem 1.1) states that

**Theorem 1.1.** There exists an absolute constant $c > 0$ such that if $X$ is an isotropic log-concave random vector in $\mathbb{R}^n$, then for every $t \geq 1$,

$$\mathbb{P}(|X| \geq ct\sqrt{n}) \leq \exp(-t\sqrt{n}).$$

This result had a huge impact on the study of log-concave measures and has a lot of applications in that subject.

A Borel probability measure on $\mathbb{R}^n$ is called log-concave if for all $0 < \theta < 1$ and all compact sets $A, B \subset \mathbb{R}^n$ one has

$$\mu((1-\theta)A + \theta B) \geq \mu(A)^{1-\theta} \mu(B)^{\theta}.$$

We refer to [5, 6] for a general study of this class of measures. Clearly, the affine image of a log-concave probability is also log-concave. The Euclidean norm of an $n$-dimensional log-concave random vector has moments of all orders (see [5]). A log-concave probability is supported on some convex subset of an affine subspace where it has a density. In particular when the support of the probability generates the whole space $\mathbb{R}^n$ (in which case we talk, in short, about full-dimensional probability) a characterization of Borell (see [5, 6]) states that the probability is absolutely continuous with respect to the Lebesgue measure and has a density which is log-concave. We say that a random vector is log-concave if its distribution is a log-concave measure.

Let $X \in \mathbb{R}^n$, be a random vector, denote the weak $p$-th moment of $X$ by

$$\sigma_p(X) = \sup_{z \in S^{n-1}} (\mathbb{E}|\langle z, X \rangle|^{p})^{1/p}.$$

The purpose of this article is to give a short proof of the following theorem.

**Theorem 1.2.** For any log-concave random vector $X \in \mathbb{R}^n$ and any $p \geq 1$,

$$\mathbb{E}|X|^p \leq C(\mathbb{E}|X| + \sigma_p(X)),$$

where $C$ is an absolute positive constant.

This result may be deduced directly from Paouris’ work [14]. Indeed, it is a consequence of Theorem 8.2 combined with Lemma 3.9 in that paper. As formulated here, Theorem 1.2 first appeared in [3] (Theorem 2). Note that because trivially a converse inequality is valid (with constant $1/2$), Theorem 1.2 states in fact an equivalence for $(\mathbb{E}|X|^p)^{1/p}$. 

2
It is noteworthy that the following strengthening of Theorem 1.2 is still open: 
\[ (E|X|^p)^{1/p} \leq E|X| + C\sigma_p(X), \] where \( C \) is an absolute positive constant.

If \( X \) is a log-concave random vector, then so is \( \langle z, X \rangle \) for every \( z \in S^{n-1} \). It follows that there exists an absolute constant \( C' > 0 \) such that for any \( p \geq 1 \), \( \sigma_p(X) \leq C'p\sigma_2(X) \) ([5]). (In fact one can deduce this inequality with \( C' = 1 \) from [4] or from Remark 5 in [11]; see also Remark 1 following Theorem 3.1 in [2].) If moreover \( X \) is isotropic, then \( E|X| \leq (E|X|^2)^{1/2} = \sqrt{n} \) and \( \sigma_2(X) = 1 \); thus 
\[ (E|X|^p)^{1/p} \leq C(\sqrt{n} + C'p). \]

From Markov’s inequality for \( p = t\sqrt{n} \), Theorem 1.2 implies Theorem 1.1 with 
\( c = (C' + 1)eC \).

Let us recall the idea underlying the proof by Paouris. Let \( X \in \mathbb{R}^n \) be an isotropic log-concave random vector. Let \( p \sim \sqrt{n} \) be an integer (for example, \( p = \lfloor \sqrt{n} \rfloor \)). Let \( Y = PX \), where \( P \) is an orthogonal projection of rank \( p \) and let \( G \) be a standard Gaussian vector \( \text{Im} P \). By rotation invariance, \( E|Y|^p \sim E|\langle G/\sqrt{p}, Y \rangle|^p \).

If the linear forms \( \langle z, X \rangle \) with \( |z| = 1 \) had a sub-Gaussian tail behaviour, the proof would be straightforward. But in general they only obey a sub-exponential tail behaviour. The first step of the proof consists of showing that there exists some \( z \) for which \( (E|\langle z, Y \rangle|)^{1/p} \) is in fact small compared to \( E|Y| \). The second step uses a concentration principle to show that \( (E_X|\langle z, PX \rangle|^p)^{1/p} \) is essentially constant on the sphere for a random orthogonal projection of rank \( p \sim \sqrt{n} \), and thus comparable to the minimum. Thus for these good projections, one has a good estimate of \( (E|Y|^p)^{1/p} \) and the result follows by averaging over \( P \).

Our proof follows the same scheme, at least for the first step, but whereas the proof of the first step in [14] is the most technical part, our argument is very simple. Then the estimate for \( \min|z|_1 E|\langle z, Y \rangle|^p \) brings us to a minimax problem precisely in the form answered by Gordon’s inequality ([9]).

Finally we would like to note that our proof can be generalized to the case of convex measures in the sense of [5, 6]. Of course the proof is longer and more technical. We provide the details in [1].

2 Proof of Theorem 1.2

First let us notice that it is enough to prove Theorem 1.2 for symmetric log-concave random vectors. Indeed, let \( X \) be a log-concave random vector and let \( X' \) be its independent copy. By Jensen’s inequality we have for all \( p \geq 1 \),
\[ (E|X|^p)^{1/p} \leq (E|X - EX|^p)^{1/p} + |EX| \leq (E|X - X'|^p)^{1/p} + E|X|. \]

On the other hand \( E|X - X'| \leq 2E|X| \) and for \( p \geq 1 \) one has \( \sigma_p(X - X') \leq 2\sigma_p(X) \). Since \( X - X' \) is log-concave (see [8]) and symmetric, we obtain that

the symmetric case proved with a constant \( C' \) implies the non-symmetric case with the constant \( C = 2C' + 1 \).
Lemma 2.1. Let \( Y \in \mathbb{R}^q \) be a random vector. Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^q \). Then for all \( p > 0 \),
\[
\min_{|z|=1} (\mathbb{E} \langle z, Y \rangle^p)^{1/p} \leq \frac{\mathbb{E} \| Y \|^p}{\mathbb{E} \| Y \|} \mathbb{E} |Y|.
\]

Proof: Let \( r \) be the largest number such that \( r \|t\| \leq |t| \) for all \( t \in \mathbb{R}^q \). Using duality, pick \( z \in \mathbb{R}^q \) such that \( |z| = 1 \) and \( \|z\|_* \leq r \) (the dual norm of \( \| \cdot \| \)). Then \( \langle z, t \rangle \leq r \|t\| \leq |t| \) for all \( t \in \mathbb{R}^q \). Therefore, \( (\mathbb{E} \langle z, Y \rangle)^p)^{1/p} \leq r(\mathbb{E} \| Y \|^p)^{1/p} \) for any \( p > 0 \), and the proof follows from \( r \mathbb{E} \| Y \| \leq \mathbb{E} |Y| \). \( \square \)

Lemma 2.2. Let \( Y \) be a full-dimensional symmetric log-concave \( \mathbb{R}^q \)-valued random vector. Then there exists a norm \( \| \cdot \| \) on \( \mathbb{R}^q \) such that
\[
(\mathbb{E} \| Y \|^q)^{1/q} \leq 500 \mathbb{E} \| Y \|.
\]

Remark. In fact the constant 500 can be significantly improved. To keep the presentation short and transparent we omit the details.

Proof: From Borell’s characterization \( Y \) has an even log-concave density \( g_Y \). Thus \( g_Y(0) \) is the maximum of \( g_Y \). Define a symmetric convex set by
\[
K = \{ t \in \mathbb{R}^q : g_Y(t) \geq 25^{-q} g_Y(0) \}.
\]
Since clearly \( K \) has a non-empty interior, it is the unit ball of a norm which we denote by \( \| \cdot \| \). On one hand, \( 1 \geq \mathbb{P}(Y \in K) = \int_K g_Y \geq 25^{-q} g_Y(0) \text{vol}(K) \), thus
\[
\mathbb{P}(\| Y \| \leq 1/50) = \int_{K/50} g_Y \leq g_Y(0) 50^{-q} \text{vol}(K) \leq 2^{-q} \leq 1/2.
\]
Therefore \( \mathbb{E} \| Y \| \geq \mathbb{P}(\| Y \| > 1/50)/50 \geq 1/100 \). On the other hand, by the log-concavity of \( g_Y \),
\[
\forall t \in \mathbb{R}^q \setminus K \quad g_{2Y}(t) = 2^{-q} g_Y(t/2) \geq 2^{-q} g_Y(t)^{1/2} g_Y(0)^{1/2} \geq (5/2)^q g_Y(t).
\]
Therefore
\[
\mathbb{E} \| Y \|^q \leq 1 + (2/5)^q \mathbb{E} \| Y \|^q = 1 + (4/5)^q \mathbb{E} \| Y \|^q.
\]
We conclude that \( (\mathbb{E} \| Y \|^q)^{1/q} \leq 5 \) and \( (\mathbb{E} \| Y \|^q)^{1/q} / \mathbb{E} \| Y \| \leq 500 \). \( \square \)

Lemma 2.3. Let \( n, q \geq 1 \) be integers and \( p \geq 1 \). Let \( X \) be an \( n \)-dimensional random vector, \( G \) be a standard Gaussian vector in \( \mathbb{R}^n \) and \( \Gamma \) be an \( n \times q \) standard Gaussian matrix. Then
\[
(\mathbb{E} |X|^p)^{1/p} \leq \alpha_p^{-1} \left( \mathbb{E} \min_{|t|=1} \| \Gamma t \| + (\alpha_p + \sqrt{q}) \sigma_p(X) \right),
\]
where \( \| |z| \| = (\mathbb{E} \langle z, X \rangle)^{1/p} \) and \( \alpha_p \) is the \( p \)-th moment of an \( N(0, 1) \) Gaussian random variable (so that \( \lim_{p \to \infty} (\alpha_p / \sqrt{p}) = 1/\sqrt{\pi} \)).
Proof: By rotation invariance, $\mathbb{E}(|G, X|^p) = \alpha_p^p \mathbb{E}|X|^p$. Notice that

$$\sigma^2 := \sup_{||t||_* \leq 1} \mathbb{E}(|G, t|^2) = \sup_{||t||_* \leq 1} |t|^2 = \sigma^2_0(X),$$

where $||.||_*$ denotes the norm on $\mathbb{R}^n$ dual to the norm $||.||$. Denote the median of $||G||$ by $M_G$. The classical deviation inequality for a norm of a Gaussian vector ([7], [15], see also [12], Theorem 12.2) states

$$\forall s \geq 0 \quad \mathbb{P} \left( \left| ||G|| - M_G \right| \geq s \right) \leq 2 \int_{s/\sigma}^{\infty} \exp \left( -t^2/2 \right) \frac{dt}{\sqrt{2\pi}}$$

and since $M_G \leq \mathbb{E}||G||$ ([10], see also [12], Lemma 12.2) this implies

$$\left( \mathbb{E}|X|^p \right)^{1/p} = \alpha_p^{-1}(\mathbb{E}||G||^p)^{1/p} \leq \alpha_p^{-1}(\mathbb{E}||G|| + \alpha_p \sigma_p(X))$$

(cf. [13], Statement 3.1).

The Gordon minimax lower bound (see [9], Theorem 2.5) states that for any norm $||.||$

$$\mathbb{E}||G|| \leq \mathbb{E} \min_{|t|=1} ||\Gamma t|| + (\mathbb{E}|H|) \max_{|z|=1} ||z|| \leq \mathbb{E} \min_{|t|=1} ||\Gamma t|| + \sqrt{q} \sigma_p(X),$$

where $H$ is a standard Gaussian vector in $\mathbb{R}^q$. This concludes the proof. □

Proof of Theorem 1.2: Assume that $X$ is log-concave symmetric. We use the notation of Lemma 2.3 with $q$ the integer such that $p \leq q < p + 1$. We first condition on $\Gamma$. Let $Y = \Gamma^* X$. Note that $Y$ is log-concave symmetric and that

$$||\Gamma t|| = (\mathbb{E}|X|(|t, X|^p)^{1/p} = (\mathbb{E}|X|(|t, \Gamma^* X|^p)^{1/p}.$$

If $\Gamma^* X$ is supported by a hyperplane then $\min_{|t|=1} (\mathbb{E}|X|(|t, \Gamma^* X|^p)^{1/p} = 0$. Otherwise Lemma 2.2 applies and combined with Lemma 2.1 gives that

$$\min_{|t|=1} ||\Gamma t|| \leq \min_{|t|=1} (\mathbb{E}|X|(|t, \Gamma^* X|^p)^{1/p} \leq 500 \mathbb{E}|X|\Gamma^* X|.$$

By taking expectation over $\Gamma$ we get

$$\mathbb{E} \min_{|t|=1} ||\Gamma t|| \leq 500 \mathbb{E}|\Gamma^* X| = 500 \mathbb{E}|H| \mathbb{E}|X| \leq 500 \sqrt{q} \mathbb{E}|X|,$$

where $H \in \mathbb{R}^q$ is a standard Gaussian vector. Applying Lemma 2.3 we obtain

$$\left( \mathbb{E}|X|^p \right)^{1/p} \leq 500 \alpha_p^{-1} \sqrt{q} \mathbb{E}|X| + (1 + \alpha_p^{-1} \sqrt{q}) \sigma_p(X).$$

This implies the desired result, since $q \leq p + 1$ and hence $\alpha_p^{-1} \sqrt{q} \leq c$ for some numerical constant $c$ (recall that $\lim_{p \to \infty} (\alpha_p/\sqrt{p}) = 1/\sqrt{c}$). □
References


6

Radosław Adamczak,
Institute of Mathematics,
University of Warsaw,
Banacha 2, 02-097 Warszawa, Poland

Rafal Latała,
Institute of Mathematics,
University of Warsaw,
Banacha 2, 02-097 Warszawa, Poland

Alexander E. Litvak,
Dept. of Math. and Stat. Sciences,
University of Alberta,
Edmonton, Alberta, Canada, T6G 2G1.

Krzysztof Oleszkiewicz,
Institute of Mathematics,
University of Warsaw,
Banacha 2, 02-097 Warszawa, Poland

Alain Pajor,
Université Paris-Est
Équipe d’Analyse et Mathématiques Appliquées,
5, boulevard Descartes, Champs sur Marne,
77454 Marne-la-Vallée, Cedex 2, France

Nicole Tomczak-Jaegermann,
Dept. of Math. and Stat. Sciences,
University of Alberta,
Edmonton, Alberta, Canada, T6G 2G1.