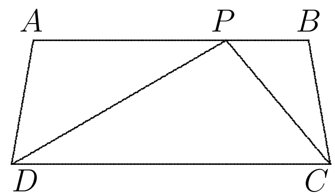
The Alberta High School Mathematics Competition
Solution to Part II, 2012.

Problem 1.
Let the dimensions of the lawn be a metres by b metres. The area of the first three rings is given by $ab-(a-6)(b-6)=6(a+b)-36$. Similarly, the area of the next four rings is given by $(a-6)(b-6)-(a-14)(b-14)=8(a+b)-160$. These two regions contain the same amount of grass and so must be the same area. Thus $6(a+b)-36=8(a+b)-160$. It follows that the only possible value of the perimeter of the lawn is $2(a+b)=124$ metres.

Problem 2.
Since $\angle DAB=\angle CBA$ and AB is parallel to DC, we have $AD=BC$. Since AB is parallel to DC, $\angle BPC=\angle PCD$. It follows that triangles BPC and PCD are similar. A similar argument shows that triangles ADP and PCD are also similar. Hence $(\frac{PD}{PC})^2=\frac{BC}{BP}\cdot\frac{AP}{AD}=\frac{PA}{PB}$, as desired.

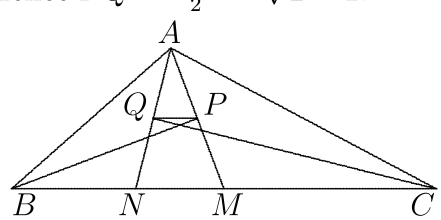


Problem 3.

(a) For any positive integer n, n^2+2 is special.

(b) We claim that for infinitely many positive integers n, n^2 is not special. Suppose $n^2=m^2+p$ for some integer m and some prime number p. Then $p=n^2-m^2=(n-m)(n+m)$. We must have $n-m=1$ and $p=n+m=2n-1$. If we let $n=3k+2$ for any positive integer k, then $2n-1=6k+3$ is not a prime number. This justifies the claim.

Problem 4.
Extend AP and AQ to cut BC at M and N respectively. Then ABM and ACN are isosceles, so that $BM=2$ and $NC=2\sqrt{2}$. Hence $MN=2\sqrt{2}-2$. Now PQ is the segment joining the midpoints of AM and AN. Hence $PQ=\frac{MN}{2}=\sqrt{2}-1$.



Problem 5.
Suppose the real numbers $x_1,\ldots,x_n$, $1\le x_i\le 4$ for $i=1,2,\ldots,n$, satisfy the two given inequalities. Then $(x_i-1)(x_i-4)\le 0$ so that $x_i+\frac{4}{x_i}\le 5$. Equality holds for $x_i=1$ or $x_i=4$. From these inequalities and the given ones, we obtain

$$5n=\frac{7n}{3}+\frac{8n}{3}\le x_1+x_2+\cdots+x_n+4\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)\le 5n.$$

Hence $x_i=1$ or 4, $i=1,2,\ldots,n$. Suppose $x_1=x_2=\cdots=x_k=1$ and $x_{k+1}=x_{k+2}=\cdots=x_n=4$ for some index k. Then $x_1+x_2+\cdots+x_n=k+4(n-k)=\frac{7n}{3}$. Hence $5n=9k$ so that 9 divides n. It follows that the smallest value of n is 9, with the numbers 1, 1, 1, 1, 1, 4, 4, 4 and 4.