The Alberta High School Mathematics Competition
Solution to Part II, 2011.

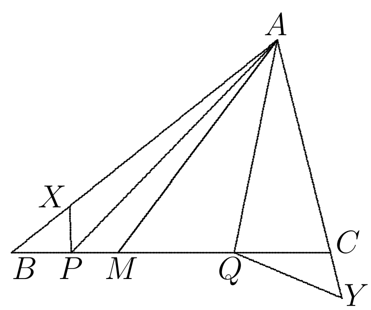
Problem 1.
The diameter of the circle, being the diagonal of a 1×3 rectangle, is $\sqrt{10}$, so the area of the circle is $\pi(\sqrt{10}/2)^2 = \frac{5\pi}{2}$. The diagonal of the square is 4, so the side of the square is $\frac{4^2}{2} = 8$. Since $\pi < 3.2 = \frac{16}{5}$, we have $\frac{5\pi}{2} < 8$. Thus the square has greater area than the circle.

Problem 2.
Eliminating z, we have $x^2 + y^2 = 2(t - x - y)$ so that $(x+1)^2 + (y+1)^2 = 2(t+1)$. In order to have a unique solution for x and y, we must have $2(t+1) = 0$ or $t = -1$.

Problem 3.
Let M be the point on PQ such that $\angle MAP = \angle BAP$. Then

$$\begin{aligned}\angle MAQ &= \angle PAQ - \angle MAP \\ &= \frac{1}{2}(\angle ABC - \angle MAB) \\ &= \frac{1}{2}\angle MAC \\ &= \angle CAQ.\end{aligned}$$

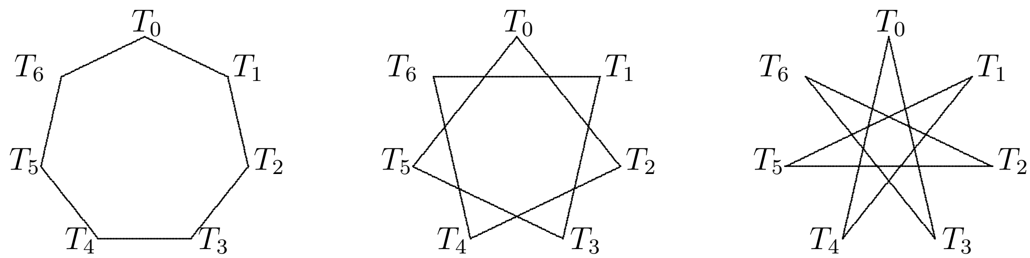
Since $\angle XPA = \angle MPA$, triangles XAP and MAP are congruent by the ASA Postulate, so that $PX = PM$. Similarly, we can prove that $QY = QM$, so that $PX + QY = PM + QM = PQ$.



Problem 4.
Since $f(n)$ takes on only positive integral values, it has a minimum value m. Let n be such that $f(n) = m$. Then $2m \leq f(n-1)+f(n+1) \leq 2f(n) = 2m$, which implies that $f(n-1) = f(n+1) = m$ also. It follows easily that $f(n) = m$ for all integers n.

Problem 5.

We first show that the conditions of the problem can be satisfied. Construct a graph where the teams are represented by vertices T_i, $0 \le i \le 6$. In the diagram below, we partition the graph into three subgraphs. Two teams play each other in the first sport if and only if the vertices representing them are joined by an edge in the first subgraph, the second sport in the second subgraph and the third sport in the third subgraph. None of the subgraphs contains a triangle.



The edges in the same subgraph have the same length, and those in different subgraphs have different lengths. In geometric terms, a diverse triple is a scalene triangle. There is basically one such triangle, namely $T_0T_1T_3$. Six others can be obtained from it by rotation, and seven more by reflection. Thus we may have as many as 14 scalene triangles.

We now prove that there are at most 14 diverse triples. Construct a complete graph on 7 vertices which represent the 7 teams. Paint an edge in the i-th colour if the teams represented by its endpoints play each other in the i-th sport, $1 \le i \le 3$. A triangle is diverse if all three sides are of different colours, and non-diverse otherwise. Since there are no monochromatic triangles, a non-diverse triangle has two sides of the same colour. Call the vertex at the junction of the two sides of the same colour its *pivot*. The number of pivots is equal to the number of non-diverse triangles. There are six edges incident with each vertex. If at least 3 of them are of the same colour, then this vertex is the pivot of at least 3 non-diverse triangles. If not, then exactly 2 edges are of each colour, so that the vertex is the pivot of exactly 3 isosceles triangles. Hence each vertex is the pivot of at least 3 non-diverse triangles. Since there are 7 vertices, this brings the total to at least 21, so that the maximum number of diverse triangles or diverse triples is 14.