# The Alberta High School Mathematics Competition
# Solution to Part II, 2010.

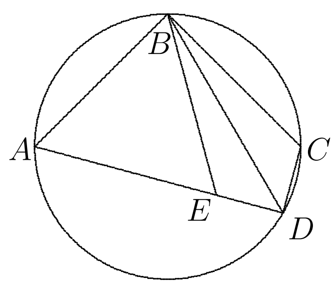
1. We have $11x + 5y + 3(100 - x - y) = 1000$ or $4x + y = 350$. Since $y \geq 0$, we get $x \leq 87$. Since $x + y \leq 100$, we also have that $3x \geq 250$, so $x \geq 84$. Thus the only solutions are $(x, y) = (84, 14)$, (85,10), (86,6) and (87,2).

2. For either (a) or (b), clearly the leading coefficient $t$ of the quadratic must be positive.

   (a) For the inequality to hold for all real $x$, the discriminant must be non-positive, that is,
   $$0 \geq (2t-1)^2 - 4t(5t-1) = 1 - 16t^2 = (1-4t)(1+4t).$$
   Since $t > 0$, $1 + 4t > 0$, so we need $1 - 4t \leq 0$. Thus $t \geq \frac{1}{4}$.

   (b) We now have the additional possibility that the two roots of the quadratic are real and non-positive. This holds if and only if $0 < t \leq \frac{1}{4}$, $2t - 1 \leq 0$ and $5t - 1 \geq 0$. This is equivalent to $\frac{1}{5} \leq t \leq \frac{1}{4}$. Combining with the answer to (a), we have $t \geq \frac{1}{5}$.

3. **First Solution:**
   Putting $AB = BC = b$ and $CD = c$, we get $AD = b + c$. Let $\angle BAD = \alpha$. Since $ABCD$ is cyclic, $\angle BCD = 180° - \alpha$. Applying the cosine law to triangles $BAD$ and $BCD$, we have $BD^2 = b^2 + (b+c)^2 - 2b(b+c)\cos\alpha$ and $BD^2 = b^2 + c^2 - 2bc\cos(180° - \alpha) = b^2 + c^2 + 2bc\cos\alpha$. Hence $b^2 + (b+c)^2 - 2b(b+c)\cos\alpha = b^2 + c^2 + 2bc\cos\alpha$, so that $b^2 + 2bc = (2b^2 + 4bc)\cos\alpha$. This yields $\cos\alpha = \frac{1}{2}$, so that $\alpha = 60°$ is the only possibility.

   **Second Solution:**
   Let $E$ be the point on $AD$ such that $DE = DC$, so that $AE = AD - DE = BC = AB$. Now $\angle BDE = \angle BDC$ since they are subtended by the equal arcs $BA$ and $BC$. It follows that triangles $BED$ and $BCD$ are congruent, so that $BE = BC = BA = AE$, triangle $BAE$ is equilateral and $\angle BAD = 60°$.

4. **First Solution:**
   The area of the punctured board is $2^{2n} - 1$. The base-2 representation of this number consists of $2n$ 1s. Since the area of each rectangle in the partition is a power of 2, we must have at least $2n$ rectangles. There exist such partitions with exactly $2n$ rectangles. Divide the board in halves by a horizontal grid line. Set aside the one with the missing square and cover the other with a rectangle of height $2^{n-1}$. Repeating the process with the strips set aside, we obtain rectangles with decreasing heights $2^{n-2}$, $2^{n-3}$, $\ldots$, $2^1$ and $2^0$, a total of $n$ rectangles. We now divide the resulting $2^n \times 1$ board in halves by a vertical line. Set aside the one with the missing square and cover the other with a rectangle of width $2^{n-1}$. Repeating the process with the strips set aside, we obtain another $n$ rectangles with decreasing widths, for a total of $2n$ rectangles in the overall partition.

**Second Solution:**

Divide the board into four congruent quadrants. Set aside the one with the missing square. Merge two of the other quadrants into one rectangle and keep the third quadrant as the second rectangle. In reducing a $2^n \times 2^n$ board down to a $2^{n-1} \times 2^{n-1}$ board, we use two rectangles. It follows that we will use exactly $2n$ rectangles in the overall partition. We now prove that we cannot get by with a smaller number. The area of a rectangle of the prescribed type is a power of 2. The smallest has area 1, and the largest has area $2^{2n-1}$. Thus there are $2n$ different sizes. If we use one of each size, the total area of these $2n$ rectangles is $1+2+\cdots+2^{2n-1} = 2^{2n}-1$, exactly the size of the punctured chessboard. Consider any other collection of rectangles whose areas are powers of 2 and whose total area is $2^{2n-1}-1$. Replace any pair of rectangles of equal area by one with twice the area. Repeat until no further replacement is possible. The resulting collection consists of rectangles of distinct areas which are powers of 2, and with total area $2^{2n-1}-1$. It can only be our collection, and since mergers only reduce the number of rectangles, $2n$ is indeed minimum.

5. (a) Note that $f(M+m)-f(m)$ is a sum of terms of the form $a_k((M+m)^k - m^k)$ where $a_k$ is the coefficient of the term $x^k$ in $f(x)$. Since each term is divisible by $M = (M+m)-m$, so is $f(M+m)-f(m)$. Since $M$ is divisible by $f(m)$, $f(M+m)-f(m)$ is divisible by $f(m)$. It follows that $f(M+m)$ is divisible by $f(m)$.

   (b) Since all the coefficients of $f$ are non-negative and $f$ is non-constant, it is strictly increasing. Let $M = f(2)f(3)$ and $n = M+2$. By (a), $f(n)$ is divisible by $f(2)$ and $f(n+1)$ is divisible by $f(3)$. Since $f(n+1) > f(n) > f(3) > f(2) > f(1) \geq 1$, both $f(n)$ and $f(n+1)$ are composite.