Alberta High School Mathematics Competition

Solutions and Comments to Part II, 2009.

Problem 1.

Since $(w+y)+(x+z)=100$, we have $w+y=50+t$ and $x+z=50-t$ for some real number t. Hence $wx+xy+yz \le (w+y)(x+z)=(50+t)(50-t)=2500-t^2 \le 2500$. This maximum value may be attained for instance when $w=x=50$ and $y=z=0$.

Problem 2.

The number of integers between a^2 and b^2 inclusive is b^2-a^2+1. The number of squares between a^2 and b^2 inclusive is $b-a+1$. From $b^2-a^2+1=100(b-a+1)$, we have $(b+a-100)(b-a)=99$. Since 99 has 6 positive divisors, there are 6 solutions, as shown in the chart below.

$b+a-100$	$b-a$	$2b-100$	b	a
1	99	100	100	1
99	1	100	100	99
3	33	36	68	35
33	3	36	68	65
9	11	20	60	49
11	9	20	60	51

Here is a slightly different approach. Let $d=b-a$. Then there are $(a+d)^2-a^2+1=2ad+d^2+1$ integers under consideration, $d+1$ of which are the squares of integers. It follows that we need $100(d+1)=2ad+d^2+1$, so that

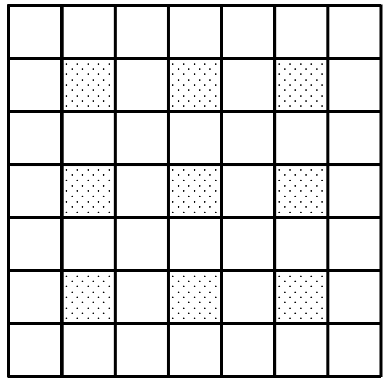
$$a=\frac{100(d+1)-d^2-1}{2d}=\frac{100-d}{2}+\frac{99}{2d}.$$

If d is even, the first term is an integer and the second is not. Hence d must be odd. Then the first term is a fraction with denominator 2, so that the second term must also be a fraction with denominator 2. This means that d must be a divisor of 99, that is, d is 1, 3, 9, 11, 33 or 99.

If $d=1$, then $a=\frac{99}{2}+\frac{99}{2}=99$ and $b=99+1=100$.
If $d=3$, then $a=\frac{97}{2}+\frac{99}{6}=65$ and $b=65+3=68$.
If $d=9$, then $a=\frac{91}{2}+\frac{99}{18}=51$ and $b=51+9=60$.
If $d=11$, then $a=\frac{89}{2}+\frac{99}{22}=49$ and $b=49+11=60$.
If $d=33$, then $a=\frac{67}{2}+\frac{99}{66}=35$ and $b=35+33=68$.
If $d=99$, then $a=\frac{1}{2}+\frac{99}{198}=1$ and $b=1+99=100$.

Problem 3.

There are 9 squares at the intersections of even-numbered rows and even-numbered columns. Any 2×2 block chosen by Betty must include one of these 9 squares. Hence Greta should play only on these squares in her first four moves. This will ensure that Betty has at most five moves, and can paint at most 20 squares brown. Hence Greta wins.



Problem 4.
Denote the circumcentre of Ω by O and note that it lies within the circle with radius $\frac{2}{3}$. We have

$$BC^2 = 4BD^2 = 4(OB^2 - OD^2) = 4\left(1 - \frac{1}{9}\right) = \frac{32}{9}$$

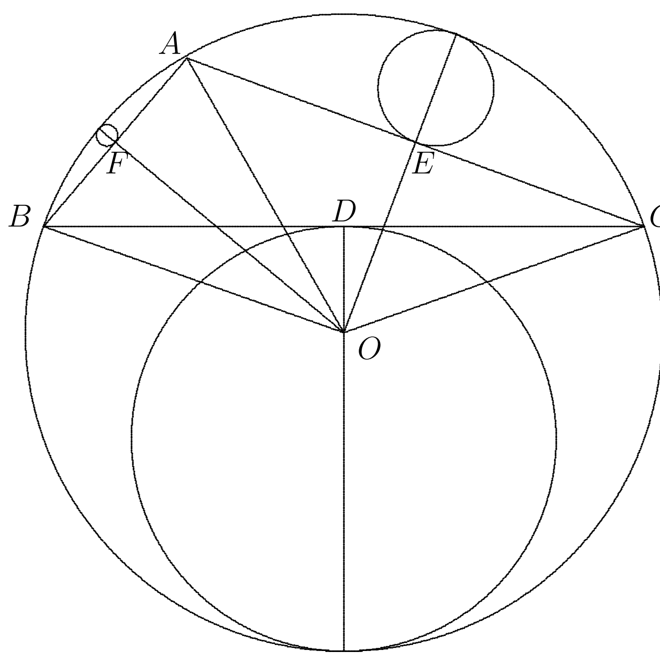
and

$$CA^2 = 4CE^2 = 4(OC^2 - OE^2) = 4\left(1 - \frac{49}{121}\right) = \frac{288}{121}.$$

By the Cosine Law, $\cos BOC = \frac{OB^2+OC^2-BC^2}{2OB\cdot OC} = -\frac{7}{9}$ and $\cos COA = \frac{OC^2+OA^2-CA^2}{2OC\cdot OA} = -\frac{23}{121}$. Hence $\sin BOC = \frac{4\sqrt{2}}{9}$ and $\sin COA = \frac{84\sqrt{2}}{121}$. It follows that

$$\cos AOB = \cos(\angle BOC - \angle COA)) = (\cos BOC)(\cos COA) + \sin BOC \sin COA = \frac{833}{1089}.$$

By the Cosine Law again, $AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos AOB = \frac{512}{1089}$. Hence we have $OF^2 = OA^2 - AF^2 = \frac{961}{1089}$ and $OF = \frac{31}{33}$. It follows that the radius of the third circle is $\frac{1}{2}(1 - \frac{31}{33}) = \frac{1}{33}$.



Problem 5.

We have $(a+1)^{a+1} = (a+1)^a(a+1) = a(a+1)^a + (a+1)^a$. Choose $a = 3^{3t}$ for an arbitrary positive integer t. Then $a = (3^t)^3$ and $(a+1)^a = ((3^{3t}+1)^{3^{3t-1}})^3$ are both cubes. If we take $k = 3^{3t}+1$, $m = 3^t(3^{3t}+1)^{3^{3t-1}}$ and $n = (3^{3t}+1)^{3^{3t-1}}$, then $k^k = m^3 + n^3$. Since t is an arbitrary positive integer, the number of possible choices for k is infinite.

Essentially the same solution is given by **Jarno Sun** of Western Canada High School, who took $k = 27n^3 + 1$ for any positive integer n. Then

$$\begin{aligned}(27n^3+1)^{27n^3+1} &= (27n^3+1)^{27n^3+1} - (27n^3+1)^{27n^3} + (27n^3+1)^{27n^3} \\ &= (27n^3+1)^{27n^3}(27n^3+1-1) + (27n^3+1)^{27n^3} \\ &= ((27n^3+1)^{9n^3}3n)^3 + ((27n^3+1)^{9n^3})^3.\end{aligned}$$

Yuri Delanghe of Harry Ainlay High School argued as follows. Let t be any integer congruent to 4 modulo 6. Then $t \equiv 1 \equiv 2^t \pmod 3$. Hence $t2^t - 1$ is divisible by 3. Let $m = n = 2^{\frac{t2^t-1}{3}}$ and $k = 2^t$. Then $m^3 + n^3 = 2^{t2^t} = (2^t)^{2^t} = k^k$.

When $t = 4$, $k = 16$, and this was also the initial example of **Danny Shi** of Sir Winston Churchill High School. He showed inductively that for any solution k, $64k$ is also a solution, in that

$$(64k)^{64k} = (4^{64k}k^{21k})^3 k^k = (4^{64k}k^{21k}a)^3 + (4^{64k}k^{21k}b)^3,$$

where $k^k = a^3 + b^3$.