Alberta High School Mathematics Competition

Solutions and Comments to Part II, 2008.

Problem 1.
We have $f(3n+1) = 3an + b + a$, $f(3n) + 1 = 3an + b + 1$ and $3f(n) + 1 = 3an + b + (2b + 1)$. Hence a, 1 and $2b+1$ must be three consecutive integers in some order. Since one of them is 1, they can only be $(-1, 0, 1)$, (0,1,2) or (1,2,3). Since $2b+1$ is odd, the second case is impossible. In the first case, we must have $2b+1 = -1$ and $a = 0$, yielding $f(n) = -1$. In the third case, we must have $2b+1 = 3$ and $a = 2$, yielding $f(n) = 2n + 1$.

Problem 2.
Let problem 1 be solved by students A, B and C. Since each pair of problems was solved by exactly one student, every other problem was solved by exactly one of A, B and C. Suppose the number of problems is at least eight. By the Pigeonhole Principle, at least three of the other seven problems were solved by one of A, B and C, say A. Let these be problems 2, 3 and 4. Then apart from A, no student solved more than one of problems 1, 2, 3 and 4. Since A did not solve all prolems, there is one, say problem 5, which A did not solve. In order to have a common solver with each of problems 1, 2, 3 and 4, problem 5 must be solved by at least four students, which is a contradiction. Hence the number of problems is at most seven. The following scheme, which has all the desired properties, shows that the number of problems can be exactly seven.

Students	Problems Solved
A	1,2,3
B	1,4,5
C	1,6,7
D	2,4,6
E	2,5,7
F	3,4,7
G	3,5,6

Problem 3.
Let the total number of canides be n initially. After Autumn has taken $\frac{na}{100} + a$ candies, Brooke will take $(n - \frac{na}{100} - a)\frac{b}{100} + b$ candies. Equating these two expressions and simplifying, we obtain $na + 100a = nb - \frac{nab}{100} - ab + 100b$. This may be rewritten as $(n+100)(b - a - \frac{ab}{100}) = 0$. Since $n \neq -100$, the second factor must be zero so that $b = \frac{100a}{100-a}$. Note that $a < 50$ so that $100 - a > 50$. Let p be any prime divisor of $100 - a$. Then p must divide $100a$ so that it divides either 100 or a. It follows that p must divide 100, so that p can only be 2 or 5. This means that $100 - a = 64$ or 80. However, 36 does not divide 6400. Hence (a, b)=(20,25) is the only possibility. This does work if we take $n = 5k$ where $3k > 40$. Autumn will take $k + 20$ candies, leaving behind $4k - 20$. Then Brooke will take $(k-5) + 25 = k + 20$ candies.

Problem 4.
Let r and s be the two real roots of $x^2 + ax + b = 0$. Then $r + s = -a$ and $rs = b$. That $(x^2 - 2cx + d)^2 + a(x^2 - 2cx + d) + b = 0$ has no real roots means that neither $x^2 - 2cx + d = r$ nor $x^2 - 2cx + d = s$ has any real roots. It follows that $(-2c)^2 < 4(d - r)$ or $c^2 < d - r$. Similarly, $c^2 < d - s$. Multiplication yields $c^4 < d^2 - d(r + s) + rs = d^2 + ad + b$.

Problem 5.

Since $AB = AC$ and $\angle CAB = 100°$, $\angle ABC = \angle BCA = 40°$. Now $\angle CAD = \angle CDA = 70°$ since $AC = DC$, so that $\angle BAD = 30°$. Since ED is parallel to AC, $\angle FDA = 70°$ and $\angle BDF = 40°$. Let P be the point on CD such that $\angle PAD = \angle FAD = 30°$. Then $\angle PAC = 40°$ so that $PA = PC$. Now triangles PAD and FAD are congruent, so that $AP = AF$. Since $\angle PAF = 60°$, PAF is an equilatral triangle. Hence $\angle APF = 60°$ so that $\angle FPD = 20°$. Moreover, $PF = PA = PC$. Hence $\angle PCF = \angle PFC = 10°$.