Alberta High School Mathematics Competition

Solutions and Comments to the Second Round, 2007.

Problem 1.

This problem really consists of two parts, finding values for n and proving that there are no more. Most contestants got somewhere with the first part, but many faltered in the second.

For $n = 1,\ 2,\ 3$ and 4, there are no positive integers m such that $m^2 + 4 \leq n$. Hence these four values have the desired property vacuously. While not an essential part of the problem, these values should be included for completeness.

If the maximum value of m is 1, then $1^2 + 4 \leq n < 2^2 + 4$ and $n = 5,\ 6$ or 7. Since 1 divides all of them, these three values have the desired property. If the maximum value of m is 2, then $2^2+4 \leq n < 3^2+4$ and $n = 8,\ 9,\ 10,\ 11$ or 12. Of these, only 8, 10 and 12 are divisible by both 1 and 2. If the maximum value of m is 3, then $3^2 + 4 \leq n < 4^2 + 4$ and $n = 13,\ 14,\ 15,\ 16,\ 17,\ 18$ or 19. Of these, only 18 is divisible by all of 1, 2 and 3. If the maximum value of m is 4, then $4^2 + 4 \leq n < 5^2 + 4$ and $n = 20,\ 21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27$ or 28. Of these, only 24 is divisible by all of 1, 2, 3 and 4. It will turn out that no positive integers other than 1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 18 and 24 have the desired property.

For the second part, **Jeffrey Mo** of William Aberhart High School argued as follows. Suppose the maximum value of m is k for some integer $k \geq 5$. Then $k^2 + 4 \leq n < (k+1)^2 + 4$. In order for n to be divisible by just $k-1$ and k, it has to be a multiple of $k(k-1)$ since $k-1$ and k are relatively prime. Now $k(k-1) < k^2 + 4$ while $2k(k-1) - ((k+1)^2 + 4) = k^2 - 4k - 5 = (k+1)(k-5) \geq 0$ for $k \geq 5$. Hence n cannot be a multiple of $k(k-1)$, so that there are no solutions for $m \geq 5$.

Jerry Lo of Ross Sheppard High School argued as follows. Suppose we have solutions n for some $m \geq 5$. Then n must be a multiple of m. Now $m^2+4 < m^2+m < m^2+2m < (m+1)^2+4 \leq m^2+3m$, with equality holding in the last case only for $m = 5$. If $n = m^2 + 3m$, then we must have $m = 5$ so that $n = 40$, but 40 is not divisible by 3. Hence $n = m(m+1)$ or $m(m+2)$. Note that n must also be a multiple of $m-1$. Note that $m-1$ and m are relatively prime. If $n = m(m+1)$, then $m-1$ must divide $m+1 = (m-1)+2$. Hence it must divide 2, so that $m \leq 3$. If $n = m(m+2)$, then $m-1$ must divide $m+2 = (m-1)+3$. Hence it must divide 3, so that $m \leq 4$. Either case contradicts $m \geq 5$. Hence there are no solutions n for $m \geq 5$.

Problem 2.

Far too many contestants did not know that the total number of tables is 15!. For those who did, the majority merely observed that the maximum sum of a row is 13+14+15=42 and the minimum sum is 1+2+3=6. From these, they concluded that the total of the two sums must be 48, in that if the maximum sum drops, the minimum sum would rise and compensate. While this may be a loose description of what is the case, it did not explain why this is the case. The argument really rests on one simple fact. The following solution by **Linda Zhang** of Western Canada High School is typical of those of the top contestants.

For each table A, there is a table B which may be obtained from A by subtracting each number in A from 16. Note that A and B are distinct tables. Now the row in A with the largest sum turns into the row in B with the smallest sum, and the row in A with the smallest sum turns into the row in B with the largest sum. The largest row sum of A plus the smallest row sum of B is 48, as is the largest row sum of B plus the smallest row sum of A. Since the 15! tables may be divided into $\frac{15!}{2}$ such pairs, the sum of the $2 \times 15!$ numbers on our record is $48 \times 15!$.

Problem 3.

This turned out to be the problem in which most contestants could make some progress. However, many approached it haphazardly, and managed to find only some of the answers. Others found all the answers but did not prove that there are no more. We give the solution by **Jarno Sun** of Western Canada High School.

Let ABC be the triangle. Let $\angle ABC = 36°$. We may assume that $\angle CAB \geq \angle BCA$. Then $\angle CAB \geq \frac{180°-36°}{2} = 72° > \angle ABC$. In order for ABC to be divided into two triangles with a straight cut, the cut must pass through a vertex. We consider three cases:

Case 1. The cut passes through B.

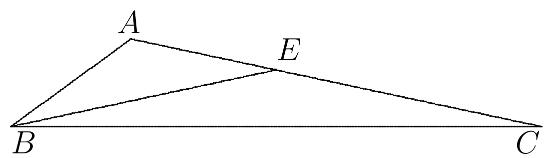
Let the cut meet CA at E. Since $\angle CAB > \angle ABC > \angle ABE$, $\angle BEA$ must be one of the equal angles in triangle BEA. It follows that $\angle BEA$ is acute so that $\angle BEC$ is obtuse. (See the diagram below.) Let $\angle EBC = \angle BCA = x°$. Then $\angle BEA = 2x°$. We consider two subcases:

Subcase 1a. $\angle BEA = \angle CAB$.

Then $\angle ABE = 180° - 4x°$ and $36° = \angle ABC = (180° - 4x°) + x°$. This yields $x = 48$ but then $\angle ABE = -12°$. This is impossible.

Subcase 1b. $\angle BEA = \angle ABE$.

Then $\angle ABE = 2x°$ and $36° = \angle ABC = 2x° + x°$. This yields $x = 12$. It follows that ABC is a $(132°, 36°, 12°)$ triangle.



Case 2. The cut passes through A.

Let the cut meet BC at D. We consider three subcases:

Subcase 2a. $\angle BDA = \angle ABC = 36°$

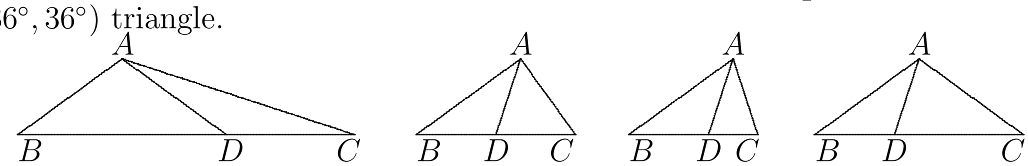
Then $\angle ADC$ is obtuse. (See the first diagram below.) We must have $\angle BCA = \angle CAD = \frac{36°}{2} = 18°$, so that ABC is a $(126°, 36°, 18°)$ triangle.

Subcase 2b. $\angle BAD = \angle ABC = 36°$.

Then $\angle BDA = 108°$. If $AD = CD$, then $\angle DAC = \angle BCA = \frac{108°}{2} = 54°$ and $\angle CAB = 90°$. (See the second diagram below.) It follows that ABC is a $(90°, 54°, 36°)$ triangle. If $AD = AC$, then $\angle BCA = \angle ADC = 72°$ and $\angle CAB = 180° - 36° - 72° = 72°$. (See the third diagram below.) It follows that ABC is a $(72°, 72°, 36°)$ triangle. Finally, if $AC = CD$, then $\angle CAD = \angle ADC = 72°$. Hence $\angle BCA = 36°$ and $\angle ABC = 108°$. (See the fourth diagram below.) It follows that ABC is a $(108°, 36°, 36°)$ triangle.

Subcase 2c. $\angle BAD = \angle BDA = 72°$.

Then $\angle ADC$ is obtuse. We must have $\angle BCA = \angle CAD = \frac{72°}{2} = 36°$ and ABC is again a $(108°, 36°, 36°)$ triangle.



Case 3. The cut passes through C.

Let the cut meet AB at F. Since $\angle CAB \geq \angle BCA > \angle ACF$, $\angle AFC$ must be one of the equal angles in triangle AFC. It follows that $\angle CFB$ is obtuse. It follows that $\angle BCF = \angle ABC = 36°$ and $\angle AFC = 72°$. Since $\angle CAB \geq \angle BCA$, ABC is again a $(72°, 72°, 36°)$ triangle.

In summary, the largest angle of ABC, namely $\angle CAB$, may be $72°$, $90°$, $108°$, $126°$ or $132°$.

Problem 4.
With greater reliance on graphing calculators and computer software, the majority of students nowadays are very uncomfortable with algebraic manipulations. A problem such as this has become inaccessible to most contestants.

Brett Baek of Western Canada High School used the following approach.

From $\frac{1-a^3}{a} = \frac{1-b^3}{b}$, we have $b - a^3b = a - ab^3$. Hence $a - b = ab(b^2 - a^2)$ so that $ab(a+b) = -1$. Similarly, $bc(b+c) = -1$. From these two equations, we have $a^2 - c^2 = bc - ab$ or $(a+c)(a-c) = -b(a-c)$. Since $a \neq c$, we have $a+b+c = 0$. Hence $abc(a+b) = -c = a+b$. Since $a+b+c = 0$ but $c \neq 0$, $a+b \neq 0$ and we have $abc = 1$. Now

$$\begin{aligned} 0 &= (a+b+c)^3 \\ &= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3 \\ &= (a^3+b^3+c^3) + 3(a+b+c)(bc+ca+ab) - 3abc \\ &= (a^3+b^3+c^3) + 0 - 3. \end{aligned}$$

It follows that the only possible value of $a^3+b^3+c^3$ is 3.

For those with more knowledge of algebra, this problem was practically trivial. **Jerry Lo** had the most succinct write-up.

The given conditions show that a, b and c are roots of the equation $x^3 + kx - 1 = 0$ where k is the common value of the three fractions. Hence

$$\begin{aligned} x^3 + kx - 1 &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (bc+ca+ab)x - abc. \end{aligned}$$

It follows that $a+b+c = 0$, $abc = 1$ and $a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-bc-ca-ab)+3abc = 3$.

Problem 5.
This problem, which looks deceptively easy, is very annoying. Only one contestant gave a complete argument, and a handful of others came close. There is a relatively easy part of the problem that many contestants got, that is, showing that $c = 50$. Suppose z people in all responded to the survey. Then $\frac{z}{2006} = \frac{c}{100}$ or $50z = 1003c$. Since 50 and 1003 are relatively prime, c must be a multiple of 50. Since we are given that $0 < a < c < b < 100$, the only possible value is $c = 50$. After this, things get messy. What follows is the approach by **Jeffrey Mo**.

Let the total number of teachers be d and the number of those teachers who responded be x. Then $\frac{x}{d} = \frac{a}{100}$ and $\frac{1003-x}{2006-d} = \frac{b}{100}$. From the first, we have $ad = 100x$. From the second, we have $2006b - bd = 100300 - 100x = 100300 - ad$. This may be rewritten as $(b-a)d = 2006(b-50)$. It follows that $1003 = 17 \times 59$ divides $(b-a)d$. Now $b - a < 100 < 1003$ and we also have $d < 1003$ since $a < c = 50$. Hence there are two cases.
Case 1. 59 divides $b - a$.
This means of course that $b - a = 59$ and $d = 2 \times 17(b-50)$. Hence $50x = \frac{ad}{2} = 17a(a+9)$. Since 25 is relatively prime to 17 and to at least one of a and $a+9$, it must divide either $a < 50$ or $a + 9 < 59$. We consider three subcases.
Subcase 1a. $a = 25$.
We have $b = 25 + 59 = 84$, $x = \frac{17 \times 25(25+9)}{50} = 289$ and $d = 2 \times 17(84 - 50) = 1156$. Hence $1003 - 289 = 714$ and $2006 - 1156 = 850$. It follows that 289 of 1156 teachers and 714 of 850 students responded to the survey.

Subcase 1b. $a + 9 = 25$.
We have $a = 25 - 9 = 16, b = 16 + 59 = 75,\ x = \frac{17\times16\times25}{50} = 136$ and $d = 2 \times 17(75 - 50) = 850$. Hence $1003 - 136 = 867$ and $2006 - 850 = 1156$. It follows that 136 of 850 teachers and 867 of 1156 students responded to the survey.
Subcase 1c. $a + 9 = 50$.
We have $a = 50 - 9 = 41$ and $b = 41 + 59 = 100$. This contradicts $b < 100$ and there are no solutions in this subcase.
Case 2. 17 divides $b - a$.
Then $b - a = 17n$ where $n \le 5$. We have $nd = 2\times59(b-50)$. Hence $50nx = \frac{and}{2} = 59a(17n-50+a)$. We consider five subcases, none of which yields additional solutions.
Subcase 2a. $n = 1$.
We have $50x = 59a(a - 33)$ and 25 must divide either a or $a - 33$. The former means $a = 25$, but then $a - 33 < 0$. The latter means $a - 33 \ge 25$, but then $a \ge 58 > 50 = c$. Both lead to contradictions.
Subcase 2b. $n = 2$.
We have $100x = 59a(a-16)$ and 25 must divide either a or $a-16$. The former means that $a = 25$, but then $59a(a - 16)$ is odd. The latter means $a - 16 = 25$, but then $59a(a - 16)$ is again odd.
Subcase 2c. $n = 3$.
We have $150x = 59a(a + 1)$ and 25 must divide either a or $a + 1$. The former means that $a = 25$, but then $59a(a + 1)$ is not divisible by 3. The latter means $a + 1 = 25$ or 50. If $a = 49$, $59a(a + 1)$ is again not divisible by 3. If $a = 24$, then $x = 236$, but then $d = \frac{100\times236}{24}$ is not an integer.
Subcase 2d. $n = 4$.
We have $200x = 59a(a+18)$ and 25 must divide either a or $a+18$. The former means that $a = 25$, but then $59a(a + 18)$ is odd. The latter means $a + 18 = 25$ or 50. If $a = 7$, $59a(a + 18)$ is again odd. If $a = 32$, then $b = 32 + 68 = 100$, and this contradicts $b < 100$.
Subcase 2e. $n = 5$.
We have $250x = 59a(35+a)$ and 25 must divide either a or $35+a$. However, this means that $a \ge 25$ and $b = a + 85 > 100$, a contradiction.

A shorter approach goes as follows. Let s and t be the respective numbers of students and teachers in the survey. Then $s+t = 2006$, both $\frac{ta}{100}$ and $\frac{sb}{100}$ are integers, and $\frac{ta}{100}+\frac{sb}{100} = 1003$. Note that 5 cannot divide both s and t. If 5 does not divide s, then b must be a multiple of 25. Since $50 < b < 100$, we must have $b = 75$. If 5 does not divide t, then a must be a multiple of 25. Since $0 < a < 50$, we must have $a = 25$. If 5 does not divide either s or t, then $a = 25$ and $b = 75$ and we have $t + 3s = 4012$. Subtract from this $s + t = 2006$ and we have $2s = 2006$, so that $s = t = 1003$. However, neither $\frac{ta}{100}$ nor $\frac{sb}{100}$ is an integer. Henceforth, we assume that 5 divides exactly one of s and t. We consider two cases. Suppose 5 divides t. Then $b = 75$ and we have $ta + 75s = 2006 \times 50$. Subtract this from $75t+75s = 2006\times75$, we have $(75-a)t = 2006\times25$. Since 5 divides t, $75-a$ divides $2\times17\times59\times5$. Since $0 < a < 50$, $25 < 75 - a < 75$. Hence we must have $75 - a = 34$ or 59. If $a = 41$, both t and a are odd, and $\frac{ta}{100}$ will not be an integer. If $a = 16$, we have $t = 850$. This leads to $s = 1156$. Thus 867 students and 136 teachers responded to the survey, yielding a total of 1003, as required by $c = 50$. Finally, since $(100 - a)t + (100 - b)s = 100(s + t) - (at + bs) = 200600 - 100300 = 100300$, the only other solution is $b = 100 - 16 = 84$ and $a = 100 - 75 = 25$, as indicated above since now 5 divides s instead of t. Solving $s + t = 2006$ and $84s + 25t = 100300$, we have $s = 850$ and $t = 1156$. Thus 714 students and 289 teachers responded to the survey, again yielding the desired total of 1003.