The Alberta High School Mathematics Competition
Solution to Part I, 2012

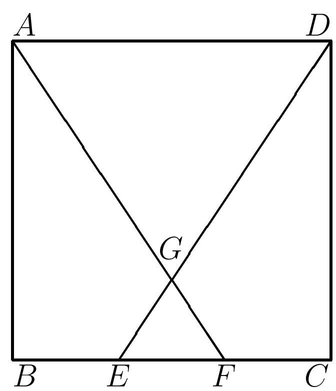
1. Note that $35 + 12 + 40 + 27 = 114$ and $1061 = 9 \times 114 + 35$. Thus Mr. Sod had spent \$35 on the tenth day of that month at pub A. The answer is **(b)**.

2. Since Meeny played 17 games, Miny and Moe played each other at most $17 + 1 = 18$ times, and each could play at most $18 + 17 = 35$ games. As Miny played 35 games, Moe did not play Meeny but played Miny 18 times. The answer is **(a)**.

3. We want $(\frac{r}{R})^2$ to be close to $\frac{1}{2}$. We have $(\frac{1}{3})^2 < (\frac{2}{4})^2 < (\frac{3}{5})^2 < (\frac{4}{6})^2 = \frac{4}{9}$ and $(\frac{5}{7})^2 = \frac{25}{49}$. Since $\frac{25}{49} - \frac{1}{2} = \frac{1}{98} < \frac{1}{18} = \frac{1}{2} - \frac{4}{9}$, the answer is **(e)**.

4. Let $f(x) = ax^2 + bx + c$. Then $1 = f(0) = c$, $0 = f(1) = a + b + c$ and $3 = f(2) = 4a + 2b + c$. We have $c = 1$, $a + b = -1$ and $2a + b = 1$. Hence $a = 2$ and $b = -3$, so that $f(3) = 10$. The answer is **(d)**.

5. Triangles GAD and GFE are similar, with $AD = 3EF$. Hence the vertical height of triangle EFG is $\frac{1}{3}$ of the vertical height of triangle ADG. Hence it is equal to $\frac{1}{4}AB = 3$ cm so that the area of triangle EFG is $\frac{1}{2} \times 3 \times 4 = 6$ cm^2. The answer is **(a)**.



6. The sum of the first n positive integers is $\frac{n(n+1)}{2}$. Suppose n is even. Then we must have either $\frac{n}{2} = 1$ or $n + 1 = 1$. Both lead to $n = 2$. Suppose n is odd. Then we must have either $\frac{n+1}{2} = 1$ or $n = 1$. However, $n = 1$ is not allowed by the hypothesis. The answer is **(b)**.

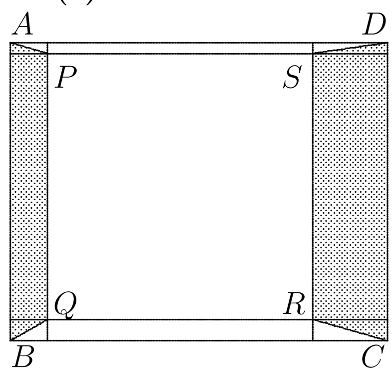
7. The fraction of problems solved only by Karla was $\frac{4}{5} - \frac{1}{2} = \frac{3}{10}$ so that the total number of problems is a multiple of 10. The fraction of problems solved by Klaus was at most $1 - \frac{3}{10} = \frac{7}{10}$. Thus the total number of problems was at least 50. If it were 50, then 10 problems were solved by Klaus alone, and as Karla solved $\frac{4}{5} \times 50 = 40$ problems, the number of problems solved by neither is 0. The total number of problems could only be as large as 70 since 35 problems would be solved by both. In this case, the number of problems solved by neither was $\frac{1}{5} \times 70 = 14$. It follows that the total number of problems must be 60, of which 30 were solved by both, 5 by Klaus alone, $\frac{3}{10} \times 60 = 18$ problems by Karla alone, and $60 - 30 - 5 - 18 = 7$ by neither of them. The answer is **(d)**.

8. Note that $a > 2$ and $a < 3$ cannot both be true as there are no integers between 2 and 3. Similarly, $a > 4$ and $a < 5$ cannot both be true. Since exactly three of the statements are true, $a < 1$ must be true. Hence the largest possible value is $a = 0$, and for this value, the three statements $a < 1$, $a < 3$ and $a < 5$ are true and the two statements $a > 2$ and $a > 4$ are false. The answer is **(a)**.

9. We have $70 = 1 \times 70 = 2 \times 35 = 5 \times 14 = 7 \times 10$. Thus there are four possible shapes of the rectangle, with respective perimeters 142 cm, 74 cm, 38 cm and 34 cm. The answer is **(d)**.

10. Solving $(n+5)^2 < 2n^2 < (n+6)^2$ yields $50 < (n-5)^2$ and $(n-6)^2 < 72$. Thus $8 \leq n-5$ and $n-6 < 9$ or $13 \leq n \leq 14$. The answer is **(c)**.

11. We shade the regions $APQB$ and $CRSD$ while leaving the regions $APSD$ and $BQRC$ unshaded. Extend the sides of $PQRS$ to the perimeter of $ABCD$, creating four rectangles at the corners each of which consists of two congruent triangles, one shaded and one unshaded. The difference between the total area of the unshaded regions (not counting $PQRS$) and the total area of the shaded regions is $1113 - 363 = 750$ cm^2. The difference in the lengths of AD and AB is 15 cm. Hence the side length of $PQRS$ is $750 \div 15 = 50$ cm, and the area of $PQRS$ is 2500 cm^2, The answer is **(c)**.



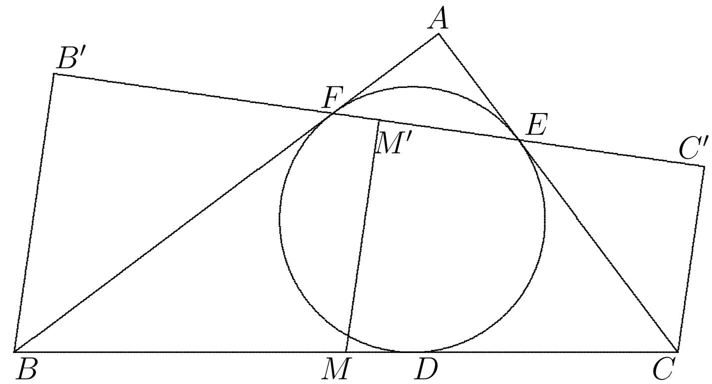
12. Let the smallest and the largest numbers Weifeng writes down be n^2 and m^2 respectively. Since they are the ends of a block of 28 consecutive numbers, $(m+n)(m-n) = m^2 - n^2 = 27$. We may have $m+n = 27$ and $m-n = 1$, whereby $m = 14$ and $n = 13$. We may have $m+n = 9$ and $m-n = 3$, whereby $m = 6$ and $n = 3$. Thus the smallest number Weifeng writes down may be $3^2 = 9$ or $13^2 = 169$. The answer is **(e)**.

13. We have

$$\begin{aligned}
\frac{1}{8} &= \frac{1}{a^2+4b+4} + \frac{1}{b^2+4a+4} \\
&= \frac{1}{(a-2)^2+4a+4b} + \frac{1}{(b-2)+4b+4a} \\
&\leq \frac{1}{4(a+b)} + \frac{1}{4(a+b)} \\
&= \frac{1}{2(a+b)}.
\end{aligned}$$

Hence $a+b \leq 4$. This maximum value is attained if and only if $a = b = 2$. The answer is **(d)**.

14. Let M be the midpoint of BC and D the point where the circle is tangent to BC. Let B', C' and M' be the respective projections of B, C and M on EF. Now AEF is a right isosceles triangle. Hence so are $BB'F$ and $CC'E$. Hence $BB' = \frac{BF}{\sqrt{2}}$ and $CC' = \frac{CE}{\sqrt{2}}$, so that $MM' = \frac{1}{2}(BB' + CC') = \frac{1}{2\sqrt{2}}(BF + CE) = \frac{1}{2\sqrt{2}}(BD + CD) = \frac{BC}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$. The answer is **(a)**.



15. Let the numbers of robots and androids be r and a respectively. After one month, these numbers became $r+7a$ and $7r-a$. After another month, they became $(r+7a)+7(7r-a) = 50r$ and $7(r+7a)-(7r-a) = 50a$. Hence after a two-month period, the number of robots became 50 times the original number, and the same goes for the number of androids. There being 6 two-month periods in a year, the initial number of robots was $46875000000 \div 50^6 = 3$ and the initial number of androids was $15625000000 \div 50^6 = 1$. The answer is **(a)**.

16. Since 13 divides $6(m+11n) = (6m+n) + 13(5n)$, 13 divides $6m+n$. Since 11 divides $6(m+13n) = (6m+n)+11(7n)$, 11 also divides $6m+n$. Hence $11 \times 13 = 143$ divides $6m+n$, so that $6m+n = 143k$ for some integer k. Since $6(m+n) = 143k+5n = 6(24k+n)-(k+n)$, 6 divides $k+n$ so that $k+n \geq 6$. Now $6(m+n) = 143k+5n = 138k+5(k+n) \geq 138+30 = 168$. Consequently $m+n \geq 28$, and this is attained if $m = 23$ and $n = 5$. The answer is **(c)**.