The Alberta High School Mathematics Competition
Solution to Part I, 2011

1. We have $2^{2012} + 4^{1006} = 2^{2012} + 2^{2012} = 2^{2013}$. The answer is **(a)**.

2. The surface area of a mini-marshmallow is 6 square centimetres while that of a giant marshmallow is 54 square centimetres. Thus the desired number of mini-marshmallows is $54 \div 6 = 9$. The answer is **(c)**.

3. Suppose there are m customers on Monday. Then there are $1.2m$ on Tuesday and $1.5m$ on Wednesday. The increase of $0.3m$ from Tuesday to Wednesday is 25% of $1.2m$. The answer is **(b)**.

4. Sawa is 2 kilometres south and 2 kilometres west of S. Her distance from S, by the Pythagorean Theorem, is $\sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$ kilometres from S. The answer is **(b)**.

5. Since $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$, the digits can only be 1, 2, 4, 5 and 8. Three of them must be 5s and they can be placed among the six digits in $\binom{6}{3} = 20$ ways. The product of the other three digits is 8, and they are (1,1,8), (1,2,4) or (2,2,2). They can be placed in 3, 6 and 1 ways respectively. Hence the total number of six-digit millenium numbers is 20(3+6+1)=200. The answer is **(e)**.

6. We have $x_3 = 24,\ x_4 = 28,\ x_5 = 26$ and $x_6 = 27$. The answer is **(d)**.

7. Squaring both sides of $2x^2 - 2x - 1 = 2x\sqrt{x^2-2x}$, we have $4x^4 - 8x^3 + 4x + 1 = 4x^4 - 8x^3$, which simplifies to $4x+1 = 0$. Hence the only solution is $x = -\frac{1}{4}$. Indeed, $2(-\frac{1}{4})^2 - 2(-\frac{1}{4}) = \frac{5}{8}$ and $2(-\frac{1}{4})\sqrt{(-\frac{1}{4})^2 - 2(-\frac{1}{4})} + 1 = \frac{5}{8}$. The answer is **(b)**.

8. Note that $g(x) = f(x+1) - f(x)$ is a linear polynomial. Since $g(1) = f(2) - f(1) = 2$ and $g(2) = f(3) - f(2) = 4$, we have $g(3) = 6$. Hence $f(4) = f(3) + g(3) = 8 + 6 = 14$. The answer is **(b)**.

9. Since a lucky number n is divisible by 7, it has the form $n = 7k$ for some positive integer k. If k is not a prime number, then it has a divisor h where $1 < h < k$, and $7h$ is a divisor of n larger than 7 but not equal to n. Hence k must be a prime number. Moreover, it cannot be greater than 7. Hence there are only 4 lucky numbers, namely, 14, 21, 35 and 49. The answer is **(d)**.

10. Annabel spent 15 seconds on each path, and Bethany 18 seconds. On the first path, Bethany was with Annabel all 15 seconds. On the second path, Bethany joined Annabel 3 seconds late, and was with her for 12 seconds. On the third path, Bethany was with Annabel for 9 seconds. On the fourth path, Bethany was with Annabel for 6 seconds. The total is 15+12+9+6=42 seconds. The answer is **(b)**.

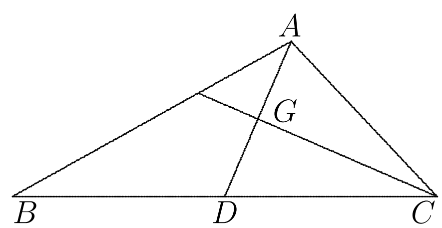
11. We can draw a regular polygon of any number of sides such that the side length is 20 centimetres. We can then draw a regular polygon of the same number of sides but with side length 15 centimetres, placed centrally inside the larger polygon. Then a tile can be chosen which can pave the ring-shaped region inside the larger polygon but outside the smaller one. Hence the answer is **(e)**.

12. Let $x_1, \ldots, x_{20}$ be the given numbers. If d is the greatest common divisor of these numbers then
$$x_1 + \cdots + x_{20} = d\left(\frac{x_1}{d} + \cdots + \frac{x_{20}}{d}\right) = 462 = 21 \cdot 22.$$
The value $d = 22$ is obtained if $x_1 = \cdots = x_{19} = d$ and $x_{20} = 2d$. For each i, $\frac{x_i}{d} \geq 1$. Hence $d \leq \frac{462}{20} = 23.1$. Since d divides 462, the largest value for d is indeed 22. The answer is **(b)**.

13. Dividing throughout by y, we have $(z^2+1)^3 > m(z^3+1)^2$ where $z = \frac{x}{y}$. This is equivalent to
$$(1-m)z^6 + 3z^4 + (3-2m)z^3 + 1 - m > 0$$
for any positive real z. Hence it is necessary to have $1 - m \geq 0$, i.e. $m \leq 1$. If we take $m = 1$, the inequality $(x^2+y^2)^3 \geq (x^3+y^3)^2$ is equivalent to
$$x^2y^2((x-y)^2 + 2x^2 + 2y^2) > 0,$$
which is clearly true. The answer is **(c)**.

14. There are $\binom{49}{2} = 1176$ colourings. The number of symmetrical colourings with respect to the middle square is $\frac{49-1}{2} = 24$. These colourings are counted twice. All the other colourings are counted four times. The desired number is $\frac{24}{2} + \frac{1176-24}{4} = 300$. The answer is **(d)**.

15. Let AD intersect the bisector of $\angle C$ at G. Then $\angle CGA = 90^\circ = \angle CGD$, $\angle GCA = \angle GCD$ and $GD = GD$. Hence triangles GCA and GCD are congruent, so that $AC = DC$. It follows that we have $BC = 2DC = 2AC$. Now among three consecutive positive integers, one is double another. This is only possible if the integers are 1, 2 and 3, or 2, 3 and 4. The former does not yield a triangle. Hence $AC = 2$, $AB = 3$ and $BC = 4$, so that $AB \cdot BC \cdot CA = 24$. The answer is **(a)**.



16. Note that if we write a positive integer m in base 3, then the base 3 representation of $\lfloor \frac{m}{3} \rfloor$ is simply the base 3 representation of m with the rightmost digit removed. Also, a positive integer m is divisible by 3 if and only if the rightmost digit of m is 0. Hence, in order that none of a_1, a_2, a_3 and a_4 is divisible by 3, the rightmost four digits of the base 3 representation of n are all non-zero. Note that $1000 > 2(3^5 + 3^4 + 3^3 + 3^2 + 3 + 1)$. If n has at most 6 digits in its base 3 representation, the first two can be any of 0, 1 and 2, while the last four cannot be 0. There are $3^2 \times 2^4 = 144$ such numbers. Clearly, n cannot have more than 7 digits as otherwise $n \geq 3^7 > 1000$. Suppose n has exactly 7 digits. As before, the last four cannot be 0. Since $1000 < 3^6 + 3^5 + 3^3 + 3^2 + 3 + 1$, the first one must be 1, the second must be 0 and the third can be any of 0, 1 and 2. Hence, there are $3 \times 2^4 = 48$ such numbers. The total is 192. The answer is **(b)**.