Alberta High School Mathematics Competition

Solution to Part I – 2009

1. If $2^x = 3^y$, then $4^x = (2^x)^2 = (3^y)^2 = 9^y$. The answer is **(d)**.

2. The number of bones Caroline bought is a multiple of 8 but 2 less than a multiple of 7. The answer is **(d)**.

3. The calculation is $\frac{(n+1)+(n+2)+\cdots+(n+100)}{100} - n = \frac{100n+(1+2+\cdots+100)}{100} - n = \frac{1+2+\cdots+100}{100}$, with $n = 0$ for Ace, $n = 1000$ for Bea and $n = 1000000$ for Cec. The answer is **(e)**.

4. Almost the entire gymnasium floor may be divided into 2×3 non overlapping rectangles each with exactly one non-blank square at the lower left corner. The answer is **(c)**.

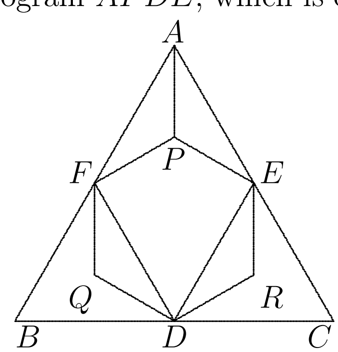
5. Observe that $x^2 - y^2 = (x - y)(x + y)$ is the product of two integers of same parity. Hence $x^2 - y^2$ is either odd or divisible by 4. Thus a number which is neither odd nor divisible by 4 cannot be expressed as a difference of two squares. On the other hand, if n is odd, then $n = 2k + 1 = (k + 1)^2 - k^2$. If n is divisible by 4, then $n = 4k = (k + 1)^2 - (k - 1)^2$. Between 1 and 2009 inclusive, there are 1005 numbers that are odd and 502 that are divisible by 4. The answer is **(d)**.

6. We can choose any six of the nine non-zero digits. The number of choices is $\binom{9}{6} = 84$. Each choice gives rise to a unique number. The answer is **(b)**.

7. Assume that the first A appears before the first B, and the first B before the first C. Then we must start with AB and continue with A or C. If we continue with A, the last three letters must be CBC. If we start with ABC, we must continue with A or B. In either case, either of the last two letters can appear before the other. So the total is $1 + 2 \times 2 = 5$. Relaxing the order of appearance, the total becomes $5 \times 3! = 30$. The answer is **(a)**.

8. Since $10^{n-1} < 2^{2009} < 10^n$ and $10^{m-1} < 5^{2009} < 10^m$, we have $10^{m+n-2} < 2^{2009}5^{2009} < 10^{m+n}$. It follows that $10^{m+n-1} = 2^{2009}5^{2009} = 10^{2009}$. Hence $m + n - 1 = 2009$. The answer is **(d)**.

9. Let ABC be the triangle and $DREPFQ$ be the hexagon, as shown in the diagram below. Triangles APE, APF, ERD and FQD are all congruent to one another. Hence $DREPFQ$ has the same area as the parallelogram $AFDE$, which is one half of $2\sqrt{3}$. The answer is **(c)**.



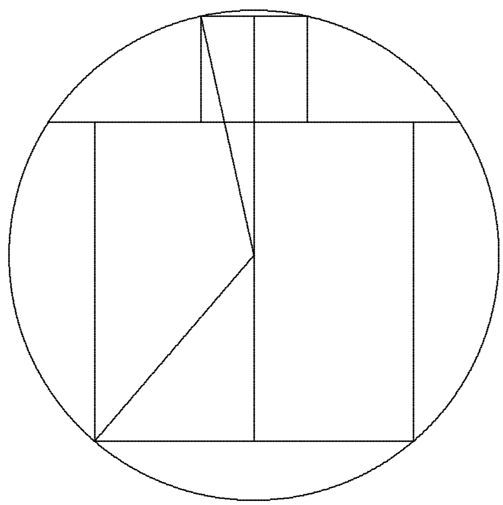
10. By the Triangle Inequality, the third side must be from 7 to 15. Now $9^2 < 11^2 - 5^2 < 10^2$, so that (5,7,11), (5,8,11) and (5,9,11) are obtuse triangles. Also, $12^2 < 11^2 + 5^2 < 13^2$, so that (5,11,13), (5,11,14) and (5,11,15) are obtuse triangles. The other three are not. The answer is **(c)**.

11. Since $Q(9) = 2009$, $x - 9$ is a factor of $Q(x) - 2009$. Since $Q(x)$ has integral coefficients, $p - 9$ divides $Q(p) - 2009 = 392 - 2009$. Since this number is odd, $p - 9$ must be odd. Since p is prime, we must have $p = 2$. Now $392 - 2009 = -1617 = -231 \times 7$, so that $Q(x)$ does exist. For instance, we may have $Q(x) = 231x - 70$. The answer is **(a)**.

12. Let 5 centimetres be the height of the parallelogram. Its base is a segment intercepted by the other pair of parallel lines 8 centimetres apart. Hence its length is at least 8 centimetres and can be arbitrarily large. The answer is **(b)**.

13. With one square root sign, $\sqrt{n} < 10$ is equivalent to $n < 100$. With two square root signs, $\sqrt{n + \sqrt{n}} < 10$ is equivalent to $n + \sqrt{n} < 100$, which is in turn equivalent to $n < 91$. With three square root signs and $n < 91$, we have $\sqrt{n + \sqrt{n + \sqrt{n}}} < \sqrt{n + 10} \leq 10$. With more square root signs, the same inequality will hold. There are 90 positive integers which satisfy $n < 91$. The answer is **(b)**.

14. Let x be the distance from the centre of the circle to the bottom edge of the larger square. The square of the radius of the circle is given by $6^2 + x^2 = 2^2 + (4 + 12 - x)^2$. This yields $x = 7$ so that the radius of the circle is $\sqrt{6^2 + 7^2} = \sqrt{85}$. The answer is **(d)**.



15. If $p = 2$, $2^p + p^2 = 8$ is not prime. If $p = 3$, $2^p + p^2 = 17$ is prime. If $p > 3$, then $p = 6k \pm 1$ for some integer k, so that $p^2 = 36k^2 \pm 12k + 1$ is 1 more than a multiple of 3. On the other hand, when divided by 3, successive powers of 2 leave remainders of 2 and 1 alternately. Since $p > 3$ is odd, 2^p is 2 more than a multiple of 3. Hence $2^p + p^2$ is divisible by 3, and cannot be prime. The answer is **(b)**.

16. Let the other root be t. Then $-a = t + 2 - \sqrt{99}$. Now $b = t(2 - \sqrt{99}) < 0$, so that $t = \sqrt{99} - 2 - a > 0$. Since a is an integer, $a \leq 9 - 2 = 7$. The answer is **(c)**.