Alberta High School Mathematics Competition

Solution to Part I – 2007

1. The number 111111 is divisible by 1001. Now $1001 = 6\times 166+5$. Hence the desired remainder is the same when we divide 11111 by 1001, which is 100. The answer is **(d)**.

2. Tell the cats to put their front legs up. Now there are still 34 heads, but only $34\times 2 = 68$ legs on the ground. Hence $80 - 68 = 12$ legs are up in the air, and each cat puts up 2 of them. It follows that the number of cats is $12 \div 2 = 6$. The answer is **(a)**.

3. Since $AB \leq 1$ and $BC \leq 2$, the area of triangle ABC is at most 2. This maximum value can be attained when $AB = 1$ and $BC = 2$ and are perpendicular to each other. Now $CA = \sqrt{5}$, and we indeed have $2 \leq CA \leq 3$. The answer is **(a)**.

4. The middle symbol must be an A. The first five symbols consist of two As and three Bs, and they can be arranged in all possible ways. The last five symbols, consisting also of two As and three Bs, must be in reverse order with respect to the first five symbols. In the first five symbols, we only have to find the number of ways of choosing two of the five positions for the two As. This is given by $\binom{5}{2} = 10$. The answer is **(c)**. An exhaustive analysis also works.

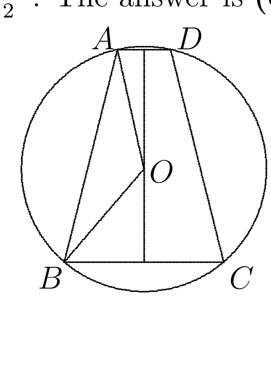
5. The integers from 11 to 30 include six primes, namely 11, 13, 17, 19, 23 and 29. In the ten odd numbers among any twenty consecutive integers each at least 7, at least three are multiples of 3 and exactly two are multiples of 5, but at most one can be a multiple of 15. Hence the maximum is indeed six. The answer is **(c)**.

6. We have $2x+2y = 5+y \geq 5$, so that the minimum value of $x+y$ is $\frac{5}{2}$, attained at $(x, y) = (\frac{5}{2}, 0)$. Also, $x + y = 5 - x \leq 5$, so that the maximum value of $x + y$ is 5, attained at $(x, y) = (0, 5)$. The answer is **(d)**.

7. If we take all the even numbers, clearly no two will differ by 3 or 5. Hence we can take at least 500 numbers. Now partition the integers from 1 to 1000 into blocks of 10. From each of the following five pairs, we can take at most one number: $(10n + 1, 10n + 4)$, $(10n + 2, 10n + 5)$, $(10n+3, 10n+8)$, $(10+6, 10n+9)$ and $(10n+7, 10n+10)$. Hence we can take no more than 500 numbers. The answer is **(d)**.

8. Suppose there exists such a polynomial p. Since $a^n - b^n$ is divisible by $a - b$ for all positive integers a, b and n with $a \neq b$, $13 - 9$ must divide $p(13) - p(9)$. However, 4 does not divide 7, and we have a contradiction. The answer is **(a)**. A parity argument also works.

9. The centre O of the circle lies on the axis of symmetry of $ABCD$. Let y be its height above BC. Then $OB^2 = y^2 + 3^2$ while $OA^2 = (8 - y)^2 + 1^2$. Equating these two values yields $y = \frac{7}{2}$. Hence the radius is $\sqrt{(\frac{7}{2})^2 + 3^2} = \frac{\sqrt{85}}{2}$. The answer is **(c)**.

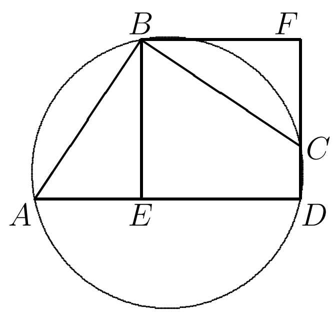


10. Let the positive integral roots be $r \leq s \leq t$. Then $x^3 - 10x^2 + ax + b = (x-r)(x-s)(x-t)$. Expansion yields $x^3 - (r+s+t)x^2 + (st+tr+rs)x - rst$. Hence $r+s+t = 10$. The possible partitions are (1,1,8), (1,2,7), (1,3,6), (1,4,5), (2,2,6), (2,3,5), (2,4,4) and (3,3,4). The answer is **(b)**.

11. Multiplying one equation by the other, we have $xy + 4 + \frac{4}{xy} = 8$. This may be rewriten as $0 = (xy)^2 - 4xy + 4 = (xy-2)^2$. Hence $xy = 2$. The answer is **(c)**.

12. Let $s = \sin^2\theta$ and $c = \cos^2\theta$. We have $\sec^2\theta + \tan^2\theta = \frac{1+s}{c} = 2$. Since $s + c = 1$, $2 - c = 2c$ so that $c = \frac{2}{3}$. It follows that $s = \frac{1}{3}$. Now $\csc^2\theta + \cot^2\theta = \frac{1+c}{s} = 5$. The answer is **(d)**.

13. Let E be the point on AD such that BE is perpendicular to AD. Complete the rectangle $BEDF$. Now $AB = BC$, $\angle AEB = 90^\circ = \angle CFB$ and $\angle ABE = 90^\circ - \angle CBE = \angle CBF$. Hence ABE and CBF are congruent triangles and they have equal area. It follows that $BEDF$ is a square, and its area is also 16. Hence $BE = 4$. The answer is **(c)**.



14. The number of incorrect answers for each of questions 1 and 6 is 0 or 5. The number of incorrect answers for each of questions 2 and 5 is 1 or 4. The number of incorrect answers for each of questions 3 and 4 is 2 or 3. A total of 8 incorrect answers can only be made up from 0+1+3+3+1+0. However, we would have an equal number of incorrect answers of T and incorrect answers of F. Hence the total must be 9, and it can be made up from either 0+1+2+2+4+0 or 0+4+2+2+1+0. However, the latter yields more incorrect answers of F than incorrect answers of T. It follows that the correct answers for the six questions are F, F, F, T, F and T respectively. Only student #4 has both an incorrect answer of T (for question 5) and an incorrect answer of F (for question 4). The answer is **(d)**.

15. We have $10^{99} \times \frac{5}{9} \leq n \leq (10^{100} - 1)\frac{5}{9}$. Since n is an integer, $5 \times 10^{98} < n < 5 \times 10^{99}$. However, we must eliminate those values of n where $5 \times 10^{98} < n < 10^{99}$. Thus the number of acceptable values of n is about 4.5×10^{99}. Since $10^{99} \leq n \leq 10^{100} - 1$, the desired probability is very close to $\frac{1}{2}$. The answer is **(c)**.

16. We can rewrite the inequality as $(k-3)(x^4+y^4+z^4) + (y^2-z^2)^2 + (z^2-x^2)^2 + (x^2-y^2)^2 \geq 0$, from which it is clear that $k \geq 3$. The answer is **(c)**.