Alberta High School Mathematics Competition

Solution to the First Round, 2006.

1. The value of $2^4 4^8 8^{16}$ is $2^{4+8\times2+16\times3} = 2^{68}$. The answer is **(c)**.

2. There are 4 such rectangles, namely, 1×2006, 2×1003, 17×118 and 34×59. The answer is **(b)**.

3. Since $100^4 = 100000000$, (100,1) is a desired pair. In fact, m can be any positive integer up to 100, and there is a unique positive integer n such that $m^4 + n = 100000001$ for this particular m. Hence the total number of pairs is 100. The answer is **(a)**.

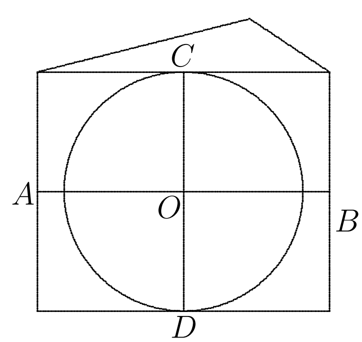
4. A and C together travel as many north-south blocks as B and D together. A and C together also travel as many east-west blocks as B and D together. Hence A and C together travel as many blocks as B and D together. It follows that the number of blocks D travels is $10 + 50 - 20 = 40$. The answer is **(d)**.

5. We are counting the number of ways of making 9 from 1, 2, 3, 4 and 5, using each number at most once. There are 3 ways, namely 1+3+5, 2+3+4 and 4+5. The answer is **(d)**.

6. Draw a circle with centre A and radius 2. Draw another circle with centre B and radius 3. The lines we seek are the common tangents of these two circles, of which there are 3. The answer is **(d)**.

7. We can steal the election by winning as little as 16 votes. We can win 2 provinces, 2 cities within each of these 2 provinces, 2 wards within each of these 4 cities, and 2 electors within each of these 8 wards. Hence $(81 - 16) + 1 = 66$ votes are required to guarantee winning the election. The answer is **(c)**.

8. Let A and B be the respective midpoints of two opposite edges of length 9. Let C and D be the respective centres of the two 7×8 faces. (See the daigram below.) Then AB and CD intersect at the centre O of the box. Place a coin of diameter 9 on the plane determined by AB and CD with centre O. Its circumference will pass through C and D, but A and B are not covered up since $AB = \sqrt{7^2 + 8^2} > 9$. Rotate the coin about AB so that it just comes off C and D. We can then expand the coin slightly and still have it fit inside the box. The answer is **(e)**. *We do not know the actual maximum value. That is why the question is phrased in its current form.*



9. Solving for b, we have $b(1-a) = 1-a$. Since a is not an integer, $1-a \neq 0$ and can be cancelled. Hence $b=1$, and is a positive integer. The answer is **(b)**.

10. Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = r$. Then $\frac{xyz}{abc} = r^3$, $x = ra$, $y = rb$ and $z = rc$. It follows that $x+y = r(a+b)$, $y+z = r(b+c)$ and $z+x = r(c+a)$, so that $\frac{x+y}{a+b} = \frac{y+z}{b+c} = \frac{z+x}{c+a} = r$. The given expression is equal to $r^3(\frac{1}{r})^3 = 1$. The answer is **(c)**.

11. We have $9\sqrt{3} - 11\sqrt{2} = (\sqrt{3} + k\sqrt{2})^3 = 3\sqrt{3} + 9k\sqrt{2} + 6k^2\sqrt{3} + 2k^3\sqrt{2}$. From $9 = 3 + 6k^2$, we have $k = \pm 1$. From $-11 = 9k + 2k^3$, $k < 0$. Hence $k = -1$. The answer is **(b)**.

12. We change the base-ten number 50 to base-two and obtain the number 110010, representing $2^5 + 2^4 + 2^2$. We now interpret this as a base-five number, representing $5^5 + 5^4 + 5^2$. Changing this number to base-ten, we obtain 3755. The answer is **(b)**.

13. Since $\frac{\sqrt{2}}{3}$ and $\frac{\sqrt{3}}{2}$ are roots, we must also have $-\frac{\sqrt{2}}{3}$ and $-\frac{\sqrt{3}}{2}$ as roots in order to have rational coefficients. Thus the degree of the polynomial cannot be less than 4, but 4 is sufficient. Of these polynomials, the one with 1 as the leading coefficient is

$$\left(x - \frac{\sqrt{2}}{3}\right)\left(x + \frac{\sqrt{2}}{3}\right)\left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right) = \left(x^2 - \frac{2}{9}\right)\left(x^2 - \frac{3}{4}\right) = x^4 - \frac{35}{36}x^2 + \frac{1}{6}.$$

Since we want integral coefficients, we clear the denominators to obtain $36x^4 - 35x^2 + 6$. Since the coefficients are relatively prime (though not pairwise relatively prime) and the smallest absolute value among the coefficients is 6, the smallest positive coefficient is also 6. The answer is **(b)**.

14. We have $a + \frac{b}{c} = 11$ and $b + \frac{a}{c} = 14$. Adding the two equations yields $(a+b)\frac{c+1}{c} = 25$ or $(a+b)(c+1) = 25c$. Since $c+1$ and c are relatively prime, $c+1$ must divide 25. Hence $c = 4$ or $c = 24$. If $c = 24$, then $a + b = 25$ but at least one of $\frac{a}{c}$ and $\frac{b}{c}$ is not an integer. Hence $c = 4$, $a + b = 20$ and $\frac{a+b}{c} = 5$. The answer is **(b)**.

15. Let the base of the pyramid be the equilateral triangle ABC. Let D be the midpoint of BC. Then $AD = \sqrt{AB^2 - BD^2} = \frac{\sqrt{3}}{2}$. Let G be the centre of ABC. Then G lies on AD and $AG = \frac{2AD}{3} = \frac{\sqrt{3}}{3}$. Let V be the fourth vertex of the pyramid and O be the centre of the sphere passing through all four vertices. Then O lies on VG and let $r = VO = OA$ be the radius of the sphere. Since $VG - VO = OG = \sqrt{OA^2 - AG^2}$, we have $1 - r = \sqrt{r^2 - \frac{1}{3}}$. Squaring both sides, we have $1 - 2r + r^2 = r^2 - \frac{1}{3}$. It follows that $r = \frac{2}{3}$. The answer is **(e)**.

16. Note that $p(-2) < p(-1) > p(0) < p(1) > p(2)$. Hence the graph of this fourth degree polynomial opens down. Let $p(x) = -x^4 + ax^3 + bx^2 + cx + d$. Then $d = p(0) = 2$. From $p(\pm 2) = 2$, we have $-16 \pm 8a + 4b \pm 2c + 2 = 2$. Hence $4b - 16 = \pm(8a + 2c)$. This is only possible if both sides are equal to 0, so that $b = 4$. Similarly, from $p(\pm 1) = 5$, we have $25a + 5c = 0$. Combining with $8a + 2c = 0$, we have $a = c = 0$, so that $p(x) = -x^4 + 4x^2 + 2 = 6 - (x^2 - 2)^2$. It follows that the maximum value of $p(x)$ is 6, occurring when $x^2 - 2 = 0$ or $x = \pm\sqrt{2}$. The answer is **(b)**.