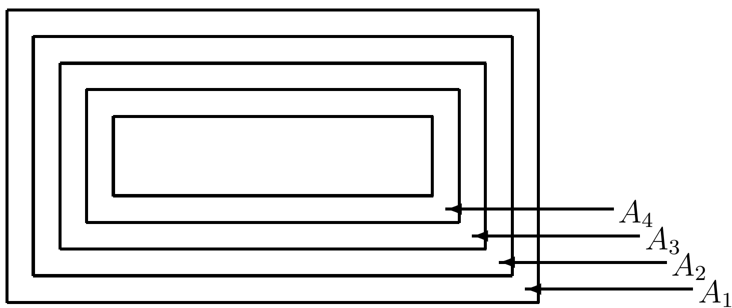
The Alberta High School Mathematics Competition
Part II, February 1, 2012.

Problem 1.
A rectangular lawn is uniformly covered by grass of constant height. Andy's mower cuts a strip of grass 1 metre wide. He mows the lawn using the following pattern. First he mows the grass in the rectangular "ring" A_1 of width 1 metre running around the edge of the lawn, then he mows the 1-metre-wide ring A_2 inside the first ring, then the 1-metre-wide ring A_3 inside A_2, and so on until the entire lawn is mowed. Andy starts with an empty grass bag. After he mows the first three rings, the grass bag on his mower is exactly full, so he empties it. After he mows the next four rings, the grass bag is exactly full again. Find, in metres, all possible values of the perimeter of the lawn.



Problem 2.
In the quadrilateral $ABCD$, AB is parallel to DC. Prove that $\dfrac{PA}{PB} = \left(\dfrac{PD}{PC}\right)^2$, where P is a point on the side AB such that $\angle DAB = \angle DPC = \angle CBA$. .

Problem 3.
A positive integer is said to be *special* if it can be written as the sum of the square of an integer and a prime number. For example, 101 is special because 101=64+37. Here 64 is the square of 8 and 37 is a prime number.

(a) Show that there are infinitely many positive integers which are special.

(b) Show that there are infinitely many positive integers which are not special.

Problem 4.
In triangle ABC, $AB = 2$, $BC = 4$ and $CA = 2\sqrt{2}$. P is a point on the bisector of $\angle B$ such that AP is perpendicular to this bisector, and Q is a point on the bisector of $\angle C$ such that AQ is perpendicular to this bisector. Determine the length of PQ.

Problem 5
Determine the smallest positive integer n for which there exist real numbers $x_1, \ldots, x_n$, $1 \le x_i \le 4$ for $i = 1, 2, \ldots, n$, which satisfy the following inequalities simultaneously:

$$x_1 + x_2 + \cdots + x_n \ge \frac{7n}{3}$$

$$\text{and } \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \ge \frac{2n}{3}.$$