Alberta High School Mathematics Competition Newsletter

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The second part of the 53rd Alberta High School Mathematics Competition was written on February 2, 2009 by 59 students representing 15 schools. Here is the list of fellowship winners and top performers.

ConocoPhillips Canada Fellows, First Places:
Danny Shi, Sir Winston Churchill High School, Calgary, and
Jarno Sun, Western Canada High School, Calgary.

Canadian Mathematical Society Fellows, Third Places:
Hunter Spink, Western Canada High School, Calgary, Grade 10, and
Noble Zhai, Western Canada High School, Calgary.

Alberta Teachers' Association Grade XI Fellow, Ninth Place:
Yishen Huang, Harry Ainlay High School, Edmonton.

Alberta Teachers' Association Grade X Fellow, Fifth Place:
Kaiven Zhou, Old Scona Academic High School, Edmonton.

Robert Barrington Leigh Memorial Fellow, Eighth Place:
Mariya Sardarli, McKernan Junior High School, Edmonton, Grade 9.

Honorable Mentions:

6	Yaroslav Babich	Sir Winston Churchill High School, Calgary, Grade 10.
7	Zili Huang	Harry Ainlay High School, Edmonton.
10	Isaac Lin	Sir Winston Churchill High School, Calgary, Grade 10.
11	Andrew Qi	Old Scona Academic High School, Edmonton, Grade 10.
12	Meng Zhao	Western Canada High School, Calgary, Grade 10, and
	Jaclyn Chang	Western Canada High School, Calgary, Grade 11.
14	Michael Wong	Western Canada High School, Calgary, Grade 11.
15	Yuri Delanghe	Harry Ainlay High School, Edmonton, Grade 11, and
	Di Mi	Western Canada High School, Calgary, Grade 11.
17	Tony Zhao	Sir Winston Churchill High School, Calgary.
18	Tim He	Henry Wisewood High School, Calgary, Grade 10.
19	Justine Zhang	Sir Winston Churchill High School, Calgary, Grade 10, and
	Michael Meiers	Old Scona Academic High School, Edmonton, Grade 11.

We offer our congratulations to the above students, their schools and their teachers.

2008 Canadian Mathematical Olympiad

1. $ABCD$ is a convex quadrilateral in which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

2. Determine all functions f defined on the set of rationals that take rational values for which $f(2f(x)+f(y)) = 2x+y$ for each x and y.

3. Let a, b and c be positive real numbers for which $a+b+c=1$. Prove that $\frac{a-bc}{a+bc}+\frac{b-ca}{b+ca}+\frac{c-ab}{c+ab} \le \frac{3}{2}$.

4. Find all functions f defined on the positive integers that take values among the positive integers for which $(f(n))^p \equiv n \pmod{f(p)}$ for all positive integers n and all prime numbers p.

5. A self-avoiding rook walk on a chessboard (a rectangular grid of squares) is a path traced by a sequence of rook moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook's path is non-self-intersecting. Let $R(m,n)$ be the number of self-avoiding rook walks on a chessboard with m rows and n columns which begin at the lower-left corner and end at the upper-left corner. For example, $R(m,1)=1$ for all positive integers m; $R(2,2)=2$; $R(3,2)=4$ and $R(3,3)=11$. Find a formula for $R(3,n)$ for each positive integer n.

Answers:

2. $f(x)=x$ or $f(x)=-x$ for all rational x.

4. $f(n)=n$ for all positive integers n; any function f with $f(p)=1$ for all primes p; any function f for which $f(2)=2$, $f(p)=1$ for primes $p>2$ and $f(n)$ and n have the same parity.

5. $R(3,n) = \frac{1}{2\sqrt{2}}((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}) - 1$.

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A Problem in Probability

A stupid king has n strong boxes constructed to store his most valuable possessions. Each strong box has a unique key. In considering what were his most valuable possessions, the king came to the conclusion that whoever got hold of the keys could help themselves to the contents of the strong boxes. It followed that his most valuable possessions were the keys. So he tossed them at random into the strong boxes, not necessarily one in each. Then he slammed the strong boxes shut, and realized what a stupid thing he had just done. To compound his error, he rashly ordered the strong boxes to be broken open, and it was only after k of them had been destroyed that he thought of using any keys he might have retrieved in the meantime. What was the probability that the king could open the remaining $n-k$ strong boxes without breaking open another one?

2008 International Mathematical Olympiad

1. Let H be the orthocenter of an acute-angled triangle ABC. The circle centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

2. (a) If $x,\ y$ and z are three real numbers, all different from 1, such that $xyz = 1$, prove that

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1.$$

   (b) Prove that equality holds for infinitely many triples of rational numbers $x,\ y$ and z.

3. Prove that there are infinite many positive integers n such that $n^2 + 1$ has a prime divisor greater than $2n + \sqrt{2n}$.

4. Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

   for all $w,\ x,\ y$ and z such that $wx = yz$.

5. Let n and k be positive integers with $k \geq n$ and $k - n$ an even number. Let $2n$ lamps labelled 1, 2, ..., $2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider the sequence of steps: at each step one of the lamps is switched (from on to off or off to on). Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on and lamps $n + 1$ through $2n$ are all off. Let M be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on and lamps $n + 1$ through $2n$ are all off, but where none of the lamps $n + 1$ through $2n$ is ever switched on. Determine $\frac{N}{M}$.

6. Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangle ABC and ADC by k_1 and k_2 respectively. Suppose there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to k_1 and k_2 intersects on k.

Answers:

4. $f(x) = x$ or $f(x) = \frac{1}{x}$.

5. 2^{k-n}.

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Answer to a Problem in Probability $\frac{k}{n}$