# Alberta High School Mathematics Competition Newsletter

**Volume 17, Number 3** **March 15, 2008**

The second part of the 52nd Alberta High School Mathematics Competition was written on February 6, 2008 by 68 students representing 21 schools. Here is the list of fellowship winners and top performers.

**ConocoPhillips Canada Fellow**, First Place:
**Jarno Sun**, Western Canada High School, Calgary, Grade 11.

**Peter H. Denham Memorial Fellow**, Second Place:
**Danny Shi**, Sir Winston Churchill High School, Calgary, Grade 11.

**Canadian Mathematical Society Fellow**, Third Place:
**Mariya Sardarli**, McKernan Junior High School, Edmonton, Grade 8.

**Alberta Teachers' Association Grade XI Fellows**, Sixth Place:
**Glen Wang**, Western Canada High School, Calgary, and
**Noble Zhai**, Western Canada High School, Calgary.

**Alberta Teachers' Association Grade X Fellows**, Fifteenth Place:
**Stephanie Bohaichuk**, Harry Ainlay High School, Edmonton, and
**Jaclyn Chang**, Western Canada High School, Calgary.

**Robert Barrington Leigh Memorial Fellow**, Fifth Place,
**Hunter Spink**, Calgary Science School, Calgary, Grade 9.

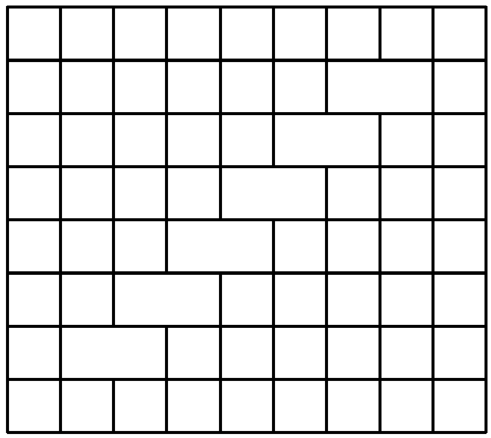
## Honourable Mentions:

| | | |
|---|---|---|
| 4 | Linda Zhang | Western Canada High School, Calgary. |
| 8 | Yu Xiang Liu | Western Canada High School, Calgary. |
| 9 | Chen Liu | Western Canada High School, Calgary, Grade 11. |
| 10 | Emma Chen | Western Canada High School, Calgary, Grade 11. |
| | Wen Wang | Western Canada High School, Calgary. |
| | Michael Zhou | Western Canada High School, Calgary. |
| 13 | Brett Baek | Western Canada High School, Calgary. |
| | Tyler Holte | Hughenden Public School, Hughenden. |
| 17 | Jacky Tian | Western Canada High School, Calgary, Grade 11. |
| | Jared Gordon | Western Canada High School, Calgary. |
| | Chong Shen | Sir Winston Churchill High School, Calgary. |
| | Frank Yang | Sir Winston Churchill High School, Calgary. |
| 21 | Di Mo | Sir Winston Churchill High School, Calgary, Grade 10. |
| 22 | Michael Wong | Tempo School, Edmonton. |

We offer our congratulations to the above students, their schools and their teachers.

**2007 Canadian Mathematical Olympiad**

1. What is the maximum number of non-overlapping $2 \times 1$ dominoes that can be placed on an $8 \times 9$ checkerboard if 6 of them are placed as shown in the diagram below. Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board.



2. You are given two similar triangles, not necessarily congruent, but with two sides of the first equal in length to two sides of the second. Prove that the ratio of the lengths of the sides that correspond under the similarity is a number between $\frac{\sqrt{5}-1}{2}$ and $\frac{\sqrt{5}+1}{2}$.

3. Suppose that $f$ is a real-valued function for which $f(xy) + f(y - x) \geq f(y + x)$ for all real numbers $x$ and $y$.

   (a) Give a non-constant polynomial that satisfies the condition.

   (b) Prove that $f(x) \geq 0$ for all real numbers $x$.

4. For two real numbers $a$ and $b$ with $ab \neq 1$, define the operation $\star$ by $a \star b = \frac{a+b-2ab}{1-ab}$. Start with a set of $n \geq 2$ real numbers whose entries $x$ all satisfy $0 < x < 1$. Select any two numbers $a$ and $b$ in the list and replace them with the number $a \star b$. Repeat this procedure until a single number remains.

   (a) Prove that this single number is the same regardless of the choice of the two numbers at each stage.

   (b) Suppose the condition on the numbers $x$ in the set is weakened to $0 < x \leq 1$. What happens if the initial set contains exactly one 1?

5. Let the incircle $\Gamma$ of triangle $ABC$ touch sides $BC$, $CA$ and $AB$ at $D$, $E$ and $F$, respectively. Let $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ denote the circumcircles of triangles $AEF$, $BFD$ and $CDE$, respectively. Let $\Gamma$ intersect $\Gamma_1$ again at $P$, $\Gamma_2$ again at $Q$ and $\Gamma_3$ again at $R$.

   (a) Prove that the circles $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ intersect in a common point.

   (b) Prove that $PD$, $QE$ and $RF$ are concurrent.

**Answers: 1.** 34. **3(a).** $x^2 + 4$. **4(b).** This single number is 1.

## 2007 International Mathematical Olympiad

1. Real numbers $a_1,\ a_2,\ \ldots,\ a_n$ are given. For each $i$, $1 \le i \le n$, define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

   and let $d = \max\{d_i : 1 \le i \le n\}$.

   (a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$, $\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}$.

   (b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that equality holds in the inequality in (a).

2. Consider five points $A,\ B,\ C,\ D$ and $E$ such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let $\ell$ be a line passing through $A$. Suppose that $\ell$ intersects the interior of the segment $DC$ at $F$ and intersects the line $BC$ at $G$. Suppose also that $EF = EG = EC$. Prove that $\ell$ is the bisector of $\angle DAB$.

3. In a mathematical competition, some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. In particular, any group of fewer than two competitors is a clique. The number of members of a clique is called its *size*. Given that in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique in one room is the same as the largest size of a clique in the other room.

4. In triangle $ABC$, the bisector of $\angle BCA$ intersects the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

5. Let $a$ and $b$ be positive integers. Prove that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

6. Let $n$ be a positive integer. Consider $S = \{(x, y, z) : x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$ as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains $S$ but does not include (0,0,0).

**Answers: 1.** (b) $x_1 = a_1 - \frac{d}{2}$ and for $2 \le k \le n$, $x_k = \max\{x_{k-1}, a_k - \frac{d}{2}\}$. **6.** $3n$.

## A Problem from Literature

Along the Grand Trunk Road in northwestern India were the cities Lahore, Umballa, Delhi, Alighur and Benares in that order. **Kim**, the title character of a novel by *Rudyard Kipling*, started from Lahore and headed for Benares. At any intermediate city, he would decide either to continue in the same direction or to turn back. His journey would come to an end if he reached Benares or returned to Lahore. If he changed directions three times altogether, his path could have taken any of the following forms:

**(1)** Lahore — Umballa — **Delhi** — **Umballa** — **Delhi** — Umballa — Lahore;
**(2)** Lahore — Umballa — Delhi — **Alighur** — Delhi — **Umballa** — **Delhi** — Umballa — Lahore;
**(3)** Lahore — Umballa — Delhi — **Alighur** — **Delhi** — **Alighur** — Delhi — Umballa — Lahore;
**(4)** Lahore — Umballa — **Delhi** — **Umballa** — Delhi — **Alighur** — Delhi — Umballa — Lahore;
**(5)** Lahore — Umballa — Delhi — **Alighur** — Delhi — **Umballa** — Delhi — **Alighur** — Delhi — Umballa — Lahore.

However, he changed directions *ten* times altogether. How many different forms could his path have taken?