The Alberta High School Mathematics Competition
Solution to Part I, November 2015.

Question 1.
How many three digit numbers have the product of the three digits equal to 5?

(a) 1 (b) 2 (c) 3 (d) 5 (e) 6

Solution:
The numbers are $115, 151, 511$. The answer is **(c)**.

Question 2.
Let m, n be positive integers such that $2^{30}3^{30} = 8^m 9^n$. Determine the value of $m + n$.

(a) 15 (b) 20 (c) 25 (d) 30 (e) 35

Solution:
The equation can be written as $2^{30-3m} = 3^{2n-30}$ hence $m = 10, n = 15$, thus $m + n = 25$. The answer is **(c)**.

Question 3.
The x-intercept, y-intercept, and slope of a certain straight line are three nonzero real numbers. The number of *negative* numbers among these three numbers is:

(a) 0 or1 (b) 1 or 2 (c) 2 or 3 (d) 0 or 2 (e) 1 or 3

Solution:
The slope of the line with the x- intercept at $(a, 0)$ and y-intercept at $(b, 0)$ is $m = -\frac{b}{a}$. If a, b are of the same sign, m is negative and if they are of opposite sign m is positive. Hence the number of negative numbers among a, b, m is 1 or 3. The answer is **(e)**.

Question 4.
The length of a certain rectangle is increased by 20% and its width is increased by 30%. Then its area is increased by:

(a) 25% (b) 48% (c) 50% (d) 56% (e) 60%

Solution:
If l and w denote the length and width of the rectangle then its area is $A = l \cdot w$ while the area of the increased rectangle is

$$(l + \frac{20l}{100}) \cdot (w + \frac{30w}{100}) = l \cdot w \cdot \frac{156}{100} = A \cdot \frac{156}{100} = A + A \cdot \frac{56}{100}.$$

Thus the area of the rectangle is increased by 56%. The answer is **(d)**.

Question 5.
Each of Alan, Bailey, Clara and Diane has a number of candies. Compared with the average of the number of candies each person has, Alan has 6 more than the average, Bailey has 2 more than the average, Clara has 10 fewer than the average and Diane has k candies more than the average. Determine k.

(a) 1 (b) 2 (c) 3 (d) 4 (e) not uniquely determined

Solution:

Let m be the average in question. Then the four people have a total of $4m$ candies. The Alan, Bailey, Clara has $m+6, m+2, m-10$ candies, which totals $3m-2$ candies. Therefore, Diane has $4m-(3m-2) = m+2$ and thus has 2 more candies than the average.

An alternate approach: since the total of the differences from the average must be zero, Diane should have just 10-6-2=2 candies more than the average.

The answer is **(b)**.

Question 6.

Ellie wishes to choose three of the seven days (Monday, Tuesday,..., Sunday) on which to wash her hair every week, so that she will never wash her hair on consecutive days. The number of ways she can choose these three days is:

(a) 6 (b) 7 (c) 8 (d) 10 (e) 14

Solution:

Ellie can choose on of the following triplets: (M, W, F), (M, W, Sa), (M, R, Sa), (T, R, Sa), (T, R, Su), (T, F, Su), (W, F, Su). There are seven possibilities.

Here is an alternate solution: In any such choice of three wash days, exactly one of them must be followed by two non-wash days. The choice of this day will determine the other two wash days. There are seven possibilities and thus the answer is **(b)**.

Question 7.

How many different sets of two or more consecutive whole numbers have sum 55?

(a) 2 (b) 3 (c) 4 (d) 5 (e) none of these

Solution:

The sum of k positive consecutive integers is

$$a+(a+1)+\cdots(a+k-1) = ka+\frac{k(k-1)}{2} = \frac{k(2a+k-1)}{2}.$$

and thus $k(2a+k-1) = 110 = 2\cdot 5\cdot 11$. The solutions (k,a) are $(2,27), (5,9)$, and $(10,1)$ for which

$$55 = 27+28 = 9+10+11+12+13 = 1+2+\cdots+10.$$

If one consider the set of whole numbers $W = \{1,2,3,...\}$ then there are three sets of consecutive whole numbers having the sum 55. However, if $W = \{0,1,2,3,...\}$ then also

$$55 = 0+1+2+\cdots+10.$$

and we find four sets with the required property.

The answer is **(b)** or **(c)**.

Question 8.

There are 5 boys and 6 girls in a class. A committee of three students is to be made such that there is a boy and a girl on the committee. In how many different ways can the committee be selected?

(a) 100 (b) 135 (c) 145 (d) 155 (e) 165

Solution:

The number of committees of 3 students made with 11 students is $\binom{11}{3} = 165$. The number of committees of three girls or three boys is $\binom{6}{3}+\binom{5}{3} = 20+10 = 30$. The number of requested committees is 165-30=135. The answer is **(b)**.

Question 9.

In a class with 20 students, 14 wear glasses, 15 wear braces, 17 wear ear-rings and 18 wear wigs. What is the minimum number of students in this class who wear all four items?

(a) 4 (b) 6 (c) 7 (d) 9 (e) 10

Solution:

We have 6 students not wearing glasses, 5 students not wearing braces, 3 students not wearing ear-rings and 2 students not wearing wigs. Even if these are 6+5+3+2=16 different students, we still have $20-16=4$ students wearing all four items. The answer is **(a)**.

Question 10.

Each person has two legs. Some are sitting on three-legged stools while the others are sitting on four-legged chairs such that all the stools and chairs are occupied. If the total number of legs is 39, how many people are there?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Solution:

Let the number of stools be m and the number of chairs be n. Then $5m+6n=39$. Hence m is a multiple of 3 but not a multiple of 6. Moreover, $5m<39$ so that $m \leq 7$. It follows that $m=3$ and $n=(39-3\times 5)\div 6=4$, so that the number of people is 3+4=7. The answer is **(c)**.

Question 11.

The number of integers n for which the fraction $\frac{2^{2015}}{5n+1}$ is an integer is

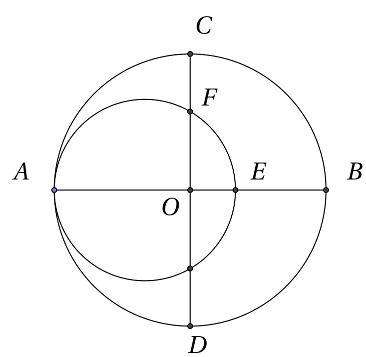
(a) 503 (b) 504 (c) 1006 (d) 1007 (e) 1008

Solution:

We should have $5n+1=\pm 2^k$ where $0 \leq k \leq 2015$. If $5n+1=2^k$, then $5|(2^k-1)$, which happens if $k=0,4,8\cdots$, hence $k=4s, 0\leq s\leq 503$. For these values of k we obtain 504 nonnegative integers n. If $5n+1=-2^k$ then $5|(2^k+1)$, which happens if $k=2,6,10,\cdots$, hence $k=4s+2$, $0\leq s\leq 503$. For these values of k we obtain 504 negative integers n. Therefore there are 1008 convenient values for n. The answer is **(e)**.

Question 12.

In the diagram below, which is not drawn to scale, the circles are tangent at A, the centre of the larger circle is at O and the lines AB and CD are perpendicular.



If $EB=3$ and $FC=2$ then the radius of the smaller circle is

(a) 4/3 (b) 5/3 (c) 5/2 (d) 3 (e) 7/2

Solution:

Let R, r denote the lengths of the large and respectively the small radius and $EB = a, FC = b$. First $a = 2R - 2r$ so that $R = r + a/2$. If O' denotes the centre of the smaller circle then $O'F^2 = O'O^2 + OF^2$ hence $r^2 = \frac{a^2}{4} + (r + \frac{a}{2} - b)^2$. Solving for r we get $r = \frac{(2b-a)^2 + a^2}{4(2b-a)}$. Taking $a = 3, b = 2$ we get $r = \frac{5}{2}$. The answer is **(c)**.

Question 13.

Consider the expansion

$$\left(1 + x + x^2 + \cdots + x^{50}\right)^3 = c_0 + c_1 x + c_2 x^2 + \cdots + c_{150} x^{150}.$$

The value of the coefficient c_{50} is

(a) 1274 (b) 1275 (c) 1326 (d) 1378 (e) none of these

Solution:

$$\left(1 + x + x^2 + \cdots + x^{50}\right)^3$$
$$= \left(1 + x + x^2 + \cdots + x^{50}\right) \cdot \left(1 + x + x^2 + \cdots + x^{50}\right) \cdot \left(1 + x + x^2 + \cdots + x^{50}\right)$$

The coefficient of x^{50} is just the number of $x^a x^b x^c = x^{a+b+c}$ with $a + b + c = 50, a, b, c \in \{0, 1, \cdots, 50\}$. If $a = 0$ the equation $b + c = 50$ has 51 solutions, namely $(0,50), (1,49), \cdots, (50,0)$. Also, if $a = 1$, the equation $b + c = 49$ has 50 solutions and so on. The number of all solutions is

$$51 + 50 + \cdots + 1 = 1326$$

Hence $c_{50} = 1326$. The answer is **(c)**.

Question 14.

A 1000 digit number has the property that every two consecutive digits form a number that is a product of four prime numbers. The digit in the 500th position is

(a) 2 (b) 4 (c) 5 (d) 6 (e) 8

Solution:

The numbers of two digits which are written as product of four prime numbers are the following: $2 \cdot 2 \cdot 2 \cdot 2 = 16$, $2 \cdot 2 \cdot 2 \cdot 3 = 24$, $2 \cdot 2 \cdot 2 \cdot 5 = 40$, $2 \cdot 2 \cdot 2 \cdot 7 = 56$, $2 \cdot 2 \cdot 2 \cdot 11 = 88$, $2 \cdot 2 \cdot 3 \cdot 3 = 36$, $2 \cdot 2 \cdot 3 \cdot 5 = 60$, $2 \cdot 2 \cdot 3 \cdot 7 = 84$, $2 \cdot 3 \cdot 3 \cdot 3 = 54$, $2 \cdot 3 \cdot 3 \cdot 5 = 90$, $3 \cdot 3 \cdot 3 \cdot 3 = 81$.

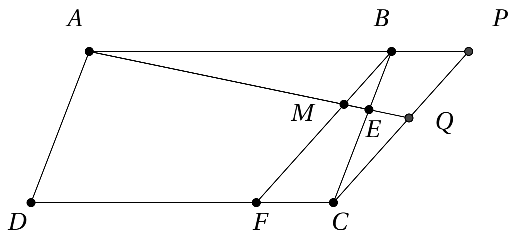
We conclude that the number that satisfies the conditions in the problem should have all its digits equal to 8. The answer is **(e)**

Question 15

Points E and F are on the sides BC and respectively CD of the parallelogram $ABCD$ such that $\frac{EB}{EC} = \frac{2}{3}$ and $\frac{FC}{FD} = \frac{1}{4}$. Let M be the intersection of AE and BF. The value of $\frac{AM}{ME}$ is equal to

(a) 11 (b) $11\frac{1}{2}$ (c) 12 (d) $12\frac{1}{2}$ (e) $12\frac{3}{4}$

Solution:



The parallel line to FB through C intercepts AB at P and AM intercepts CP at Q. Then

$$\frac{MQ}{ME} = \frac{ME+EQ}{ME} = 1+\frac{EQ}{ME} = 1+\frac{EC}{EB} = 1+\frac{3}{2} = \frac{5}{2}$$

and

$$\frac{AM}{MQ} = \frac{AB}{BP} = \frac{DC}{FC} = \frac{DF+FC}{FC} = 1+\frac{DF}{FC} = 4+1 = 5$$

Hence $\frac{AM}{ME} = \frac{AM}{MQ}\cdot\frac{MQ}{ME} = 5\cdot\frac{5}{2} = 12.5$. The answer is **(d)**.

Question 16.

Each of Alvin, Bob and Carmen spent five consecutive hours composing problems. Alvin started alone, and was later joined by Bob. Carmen joined in before Alvin stopped. When one person was working alone 4 problems were composed per hour. When two people were working together, each only composed 3 problems per hour. When all three were working, each composed only 2 problems per hour. No coming or going occurs during the composition of any problem. At the end, 46 problems were composed. How many were composed by Bob?

(a) 14 (b) 15 (c) 16 (d) 17 (e) 18

Solution:

The total work period may be divided into five intervals by comings and goings. The respective numbers of people working during these intervals are 1, 2, 3, 2 and 1 respectively. Note that the total length of any three consecutive intervals is 5 hours. Hence the fourth interval has the same length as the first and the fifth interval has the same length as the second. During each of the second, third and fourth interval, the number of problems composed was 6 per hour since 3+3=6=2+2+2. Hence $5 \times 6 = 30$ problems were composed when Bob was working. The number of problems composed when Alvin or Carmen was working alone was 46-30=16. It follows that the total length of these two intervals is equal to $16 \div 4 = 4$ hours. Hence the total length of the second and the fourth interval is also 4 hours, so that the length of the third interval is $5-4=1$ hour. Thus the number of problems composed by Bob was $1\times 2+4\times 3 = 14$. The answer is **(a)**.