The Alberta High School Mathematics Competition

Solution to Part I, 2014.

Question 1.
When the repeating decimal $0.\overline{6}$ is divided by the repeating decimal $0.\overline{3}$, the quotient is

(a) $0.\overline{2}$ (b) 2 (c) $0.\overline{5}$ (d) 0.5 (e) none of these

Solution:
We have $0.\overline{6} \div 0.\overline{3} = \frac{2}{3} \div \frac{1}{3} = 2$.The answer is **(b)**.

Question 2.
The number of two-digit positive integers such that the sum of the two digits is 12 is

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Solution:
There are 7 such numbers, namely 39, 48, 57, 66, 75, 84 and 93. The answer is **(c)**.

Question 3.
Let $a > b > 0$ be prime numbers. Of the following five numbers, the one which cannot be equal to $a - b$ is

(a) 41 (b) 42 (c) 43 (d) 44 (e) 45

Solution:
We have $43 - 2 = 41$, $47 - 5 = 42$, $47 - 3 = 44$ and $47 - 2 = 45$. In order for $a - b = 43$, we must have $b = 2$ but then $a = 45$ is not a prime number. The answer is **(c)**.

Question 4.
Chau and Matt are picking berries. On Monday Matt picks twice as many kilograms of berries as Chau does, and on Tuesday Chau picks twice as many kilograms of berries as Matt does. Between them, they pick a total of 30 kilograms of berries over the two days. The number of kilograms of berries Chau picks over the two days is

(a) 14 (b) 15 (c) 16 (d) some number greater than 16

(e) not uniquely determined

Solution:
Let Chau pick x kilograms of berries on Monday and Matt pick y kilograms of berries on Tuesday. Then $3x + 3y = 30$ so that $x + y = 10$. The number of kilograms Chau picks over the two days is $x + 2y$. This expression can take on infinitely many values if all we know is that $x + y = 10$. For example we may have $x = 0$ so that $y = 10$ and $x + 2y = 20$; or we may have $x = 4$ so that $y = 6$ and $x + 2y = 16$; and so on. The answer is **(e)**.

Question 5.
For every set of five of the numbers 1, 2, ..., 2014, Lac writes down the smallest of the five numbers. The largest number she writes down is

(a) 5 (b) 2009 (c) 2010 (d) 2014 (e) none of these

Solution:
The largest minimum occurs when Lac chooses the numbers 2010, 2011, 2012, 2013 and 2014, and writes down 2010. The answer is **(c)**.

Question 6.
The sum of four consecutive integers is n. The largest of these four numbers is

(a) n (b) $\frac{n}{4}$ (c) $\frac{n-2}{2}$ (d) $\frac{n+6}{4}$ (e) none of these

Solution:
Let $n = (k-3)+(k-2)+(k-1)+k = 4k-6$. Then $k = \frac{n+6}{4}$. The answer is **(d)**.

Question 7.
Let $f(n) = 2n^3$ for any positive integer n. For any odd number $m \geq 3$, the largest positive integer k such that 2^k divides $f(f(f(m)))$ is

(a) 4 (b) 8 (c) 12 (d) 13 (e) not uniquely determined

Solution:
We have $f(m) = 2m^3$, $f(2m^3) = 2^4m^9$ and $f(2^4m^9) = 2^{13}m^{27}$. The answer is **(d)**.

Question 8.
Let f be a function such that $f(x) + 3f(\frac{1}{x}) = x^2$ for any nonzero real number x. Then the value of $f(-3)$ is

(a) $-\frac{77}{72}$ (b) $-\frac{37}{36}$ (c) $-\frac{25}{24}$ (d) $-\frac{13}{12}$ (e) none of these

Solution:
Replacing x by -3 and then by $-\frac{1}{3}$ in the given equation, we obtain $f(-3) + 3f(-\frac{1}{3}) = 9$ and respectively $f(-\frac{1}{3}) + 3f(-3) = \frac{1}{9}$. Solving for $f(-3)$ we get $f(-3) = -\frac{13}{12}$. The answer is **(d)**.

Alternative Solution: Replacing x by $\frac{1}{x}$ in the given equation, we obtain $f(\frac{1}{x}) + 3f(x) = \frac{1}{x^2}$. Solving for $f(x)$ we get $f(x) = \frac{3-x^4}{8x^2}$. Hence $f(-3) = -\frac{39}{36} = -\frac{13}{12}$.

Question 9.
The number of integer pairs (m, n) such that $mn = m + n$ is

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

Solution:
Clearly $n \neq 1$, so that $m = \frac{n}{n-1} = 1 + \frac{1}{n-1}$. Since m is an integer, we can only have $n = 0$ or 2, with $m = 0$ and 2 respectively. The answer is **(c)**.

Question 10.
The number of five-digit positive integers such that the digits are alternately odd and even (either odd-even-odd-even-odd or even-odd-even-odd-even) is

(a) 10×5^5 (b) 2×5^5 (c) 5^5 (d) 9×5^4 (e) none of these

Solution:
For the odd-even-odd-even-odd pattern, we have 5^5 such numbers. For the even-odd-even-odd-even pattern, we have 4×5^4 such numbers. The total is 9×5^4. The answer is **(d)**.

Question 11.
Let a, b and c be real numbers such that $a + b + c = 5$ and $\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} = 6$. The value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$ is

(a) 21 (b) 23 (c) 25 (d) 27 (e) not uniquely determined

Solution:
Note that

$$\begin{aligned}\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= \frac{5-(b+c)}{b+c} + \frac{5-(c+a)}{c+a} + \frac{5-(a+b)}{a+b} \\ &= \frac{5}{b+c} + \frac{5}{c+a} + \frac{5}{a+b} - 3.\end{aligned}$$

Hence $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 5 \times 6 - 3 = 27$. The answer is **(d)**.

Question 12.
P is a point inside an acute triangle ABC. D, E and F are the feet of the perpendiculars from P on BC, CA and AB respectively. If $BD = 1$, $DC = 10$, $CE = 6$, $EA = 9$ and $AF = 13$, the length of FB is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution:
By Pythagoras' Theorem, $BD^2 - DC^2 = (BP^2 - PD^2) - (CP^2 - PD^2) = BP^2 - CP^2$. Similarly, $CE^2 - EA^2 = CP^2 - AP^2$ and $AF^2 - BF^2 = AP^2 - BP^2$. Summing these equations yields $(BD^2 + CE^2 + AF^2) - (DC^2 + EA^2 + FB^2) = 0$. Hence $FB^2 = (1^2 + 6^2 + 13^2) - (10^2 + 9^2) = 5^2$ so that $FB = 5$. The answer is **(e)**.

Question 13.
Three candles which can burn for 30, 40 and 50 minutes respectively are lit at different times. All three candles are burning simultaneously for 10 minutes, and there is a total of 20 minutes in which exactly one of them is burning. The number of minutes in which exactly two of them are burning is

(a) 35 (b) 45 (c) 70 (d) 90 (e) none of these

Solution:
Count the number of minutes in which each candle is burning and add them together. The ten minutes of simultaneous burning contributes 30 minutes to this, and there are 20 minutes of individual burning. The total must be 30+40+50=120 minutes, so the time when exactly two candles are burning must contribute the remaining 120-30-20=70 minutes. Thus the answer must be 70/2=35.

Alternative Solution: Let the first two candles burn simultaneously for a minutes, the first and the third for b minutes and the last two for c minutes. Then the first candle burns alone for $20-a-b$ minutes, the second for $30-a-c$ minutes and the third for $40-b-c$ minutes. Hence $90+2(a+b+c)=20$ so that $a+b+c=35$. This may be realized if $a=0$, $b=20$ and $c=15$. The three candles are lit at minutes 30, 0 and 15 respectively. The answer is **(a)**.

Question 14.
P is a point inside a convex quadrilateral $ABCD$ of area 168 such that $PA=9$, $PB=PD=12$ and $PC=5$. The perimeter of the quadrilateral is

(a) 38 (b) 56 (c) 58 (d) 60 (e) 62

Solution:
Note that $AC \leq AP+PC=14$, $BD \leq BP+PD=24$ and the area of $ABCD$ is at most $\frac{1}{2}AC \cdot BD \leq 168$. Thus we must have equality, so that P is the point of intersection of the diagonals which must be perpendicular to each other. By Pythagoras' Theorem, $AB=AD=\sqrt{9^2+12^2}=15$ and $CB=CD=\sqrt{5^2+12^2}=13$. Hence the perimeter of $ABCD$ is $2(15+13)=56$. The answer is **(b)**.

Question 15.
For any real numbers x and y, the minimum value of $x^4-5x^2+y^2+2x+2y+2xy+6$ is

(a) -5 (b) -4 (c) -1 (d) 0 (e) none of these

Solution:
Since $x^4-5x^2+y^2+2x+2y+2xy+6=(x^2-3)^2+(x+y+1)^2-4$, minimum value is -4, attained at $x=\pm\sqrt{3}$ and $y=-1\mp\sqrt{3}$. The answer is **(b)**.

Question 16.
Four cages are arranged in a 2×2 formation. Each cage contains some chickens and some rabbits (some cages may contain only chickens or only rabbits). The total number of heads in the two cages in the first row is 60. The total number of legs in the two cages in the second row is 240. The total number of heads in the two cages in the first column is 70. The total number of legs in the two cages in the second column is 230. The minimum number of animals in all four cages is

(a) 120 (b) 128 (c) 145 (d) 180 (e) none of these

Solution:
There are 230 legs in the two cages in the second column. Since $230=4\times 57+2$, they may come from as few as 58 animals, namely, 57 rabbits and 1 chicken. Hence the minimum number of animals in all four cages is 70+58=128. If we have 55 animals in the first cage in the first row, 0

chickens and 5 rabbits in the second cage in the first row, 15 chickens and 0 rabbits in the first cage in the second row, and 1 chicken and 52 rabbits in the second cage in the second row, we have 128 animals overall. The answer is **(b)**.