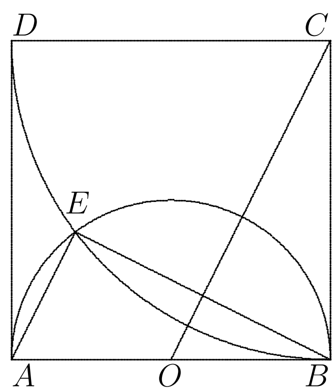
The Alberta High School Mathematics Competition

Solution to Part II, 2014

Problem 1.
$ABCD$ is a square. The circle with centre C and radius CB intersects the circle with diameter AB at $E \neq B$. If $AB = 2$, determine AE.

Solution:
Let O be the midpoint of AB. Then $OB = 1$ and $OC = \sqrt{1^2 + 2^2} = \sqrt{5}$. The line OC of centres of the two circles is perpendicular to the common chord BE. AE is also perpendicular to BE since E lies on the circle with diameter AB. Now $\angle ABE = 90° - \angle COB = \angle OCB$. Hence triangles ABE and OCB are similar, so that $AE = \frac{AB \cdot OB}{OC} = \frac{2\times1}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$.



Problem 2.
A family consists of two parents of the same age, and a number of children all of different ages. The average age of the children is 15, and the average age of the whole family is 21. When each child was born, the parents were at least 25 and at most 35 years old. All ages are given in whole numbers of years. Find all possible values of the number of children in this family.

Solution:
Let the age of the parents be p and let the number of children be n. Then the total age of the family is $15n + 2p = 21(n + 2)$, which simplifies to $p = 3n + 21$. Since the children are of different ages and their average age is 15, the age of the eldest one is at least $15 + \frac{n-1}{2} = \frac{n+29}{2}$. It follows that $3n + 21 = p \geq \frac{n+29}{2} + 25 = \frac{n+79}{2}$. This simplifies to $5n \geq 37$ which implies that $n \geq 8$. On the other hand, the age of the youngest child is at most $15 - \frac{n-1}{2} = \frac{31-n}{2}$. It follows that $3n + 21 = p \leq \frac{31-n}{2} + 35 = \frac{101-n}{2}$. This simplifies to $7n \leq 59$ which implies that $n \leq 8$. It follows that the only possible value is $n = 8$. This may be realized if the children are of ages 11, 12, 13, 14, 16, 17, 18 and 19, and both parents are of age 45.

Problem 3.
Two cars 100 metres apart are travelling in the same direction along a highway at the speed limit of 60 kph. At one point on the highway, the speed limit increases to 80 kph. Then a little later, it increases to 100 kph. Still later, it increases to 120 kph. Whenever a car passes a point where the speed limit increases, it instantaneously increases its speed to the new speed limit. When both cars are travelling at 120 kph, how far apart are they?

Solution:
Let the first car be at a point B while the second car is at a point A, both in the 60 kph zone. Then $AB = 100$ metres. Let the first car be at a point D while the second car is at a point C, both in the 120 kph zone. Now the amount of time the second car takes to go from A to C is the same as the amount of time the first car takes to go from B to D. Both cars take the same amount of time going from B to C. Hence the amount of time the second car takes to go from A to B at 60 kph is the same as the amount of time the first car takes to go from C to D at 120 kph. It follows that $CD = 2AB = 200$ metres.

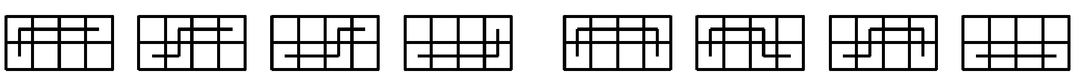
Problem 4.
Let $p(x)$be a polynomial with integer coefficients such that $p(1) = 5$ and $p(-1) = 11$.
(a) Give an example of $p(x)$ which has an integral root.
(b) Prove that if $p(0) = 8$, then $p(x)$ does not have an integrral root.

Solution:

(a) We are given two pieces of information. So we seek a polynomial with two undetermined coefficients. The first attempt is $p(x) = ax+b$. Then $5 = p(1) = a+b$ and $11 = p(-1) = -a+b$. Hence $a = -3$ and $b = 8$, but the only root of $-3x + 8 = 0$ is $x = \frac{8}{3}$, which is not integral. However, it is easy to modify our polynomial to $p(x) = 8x^2 - 3x$. We have $p(1) = 5$ and $p(-1) = 11$, but this time, we have an integral root $x = 0$ in addition to $x = \frac{3}{8}$.

(b) Suppose $p(x)$ has an integral root $x = r$. Then $r - 1$ divides $p(r) - p(1) = -5$, so that r is one of -4, 0, 2 or 6. Also, $r + 1 = r - (-1)$ divides $p(r) - p(-1) = -11$, so that r is one of -12, -2, 0 and 10. The only common value between the two lists is $r = 0$, but $p(0) = 8$. This is a contradiction.

Problem 5.
On a $2 \times n$ board, you start from the square at the bottom left corner. You are allowed to move from square to adjacent square, with no diagonal moves, and each square must be visited at most once. Moreover, two squares visited on the path may not share a common edge unless you move directly from one of them to the other. We consider two types of paths, those ending on the square at the top right corner and those ending on the square at the bottom right corner. The diagram below shows that there are 4 paths of each type when $n = 4$. Prove that the numbers of these two types of paths are the same for $n = 2014$.



Solution:
The path of the marker is uniquely determined by its vertical moves. The only condition is that no two vertical moves can be made in adjacent columns. Whether the path ends in the upper or lower right corner is determined by the parity of the number of vertical moves. Let the columns be represented by elements in the set $\{1, 2, \ldots, n\}$. Consider all subsets which do not contain two consecutive numbers. Let a_n be the number of such subsets of even size, and b_n be the number of such subsets of odd size. Then $a_0 = a_1 = a_2 = 1$ because of the empty subset, $b_0 = 0, b_1 = 1$ and $b_2 = 2$.

For $n \geq 3$, classify the subsets of $\{1, 2, \ldots, n\}$ into two types, those containing $n-1$ and those not containing $n-1$. A subset of the first type cannot contain either $n-2$ or n. Hence the number of such subsets of even size is b_{n-3} and the number of such subsets of odd size is a_{n-3}. The subsets of the second type may be divided into pairs such that in each pair, the two subsets are identical except that one contains n and the other does not. Hence the number of such subsets of even size is equal to the number of such subsets of odd size. It follows that $a_n - b_n = b_{n-3} - a_{n-3}$. Hence

$$\begin{aligned}
a_{3k} - b_{3k} &= (-1)^k(a_0 - b_0) &= (-1)^k, \\
a_{3k+1} - b_{3k+1} &= (-1)^k(a_1 - b_1) &= 0, \\
a_{3k+2} - b_{3k+2} &= (-1)^k(a_2 - b_2) &= (-1)^{k+1}.
\end{aligned}$$

In particular, $a_{2014} = b_{2014}$.