The Alberta High School Mathematics Competition
Solution to Part I, 2013.

1. Of the first 100 positive integers 1, 2, ..., 100, the number of those not divisible by 7 is

   (a) 14 (b) 15 (c) 85 (d) 86 (e) none of these

   Solution:
   When 100 is divided by 7, the quotient is 14, with a remainder of 2. Thus 14 of the first 100 positive integers are divisible by 7. It follows that the number of these integers not divisible by 7 is $100 - 14 = 86$. The answer is **(d)**.

2. The total score of four students in a test is 2013. Ace scores 1 point more than Bea, Bea scores 3 points more than Cec, and Cec scores 2 points more than Dee. None of their scores is divisible by

   (a) 3 (b) 4 (c) 5 (d) 11 (e) 23

   Solution:
   Since the average score is just over 500, we try 500 as a score for Dee. Then Cec's score is 502, Bea's is 505 and Ace's 506. The total is indeed 2013, so that no adjustment is necessary. Now 500 is divisible by 4, 500 and 505 are divisible by 5, and 506 is divisible by 11 and 23. The answer is **(a)**.

3. Of the following five fractions, the largest one is

   (a) $\frac{1}{75}$ (b) $\frac{2}{149}$ (c) $\frac{3}{224}$ (d) $\frac{4}{299}$ (e) $\frac{6}{449}$

   Solution:
   The lowest common numerator is 12. The fractions then become $\frac{12}{900}$, $\frac{12}{894}$, $\frac{12}{896}$, $\frac{12}{897}$ and $\frac{12}{898}$ respectively. The answer is **(b)**.

4. Two teams A and B played a soccer game on each of seven days. On each day, the first team to score seven goals wins. There were no ties. Over the seven days, A won more games than B, but B scored more goals than A overall. The difference in the total numbers of goals scored by B and A is at most

   (a) 17 (b) 18 (c) 19 (d) 20 (e) none of these

   Solution:
   B won at most three games, and for each of these games, B wins by at most 7 goals with a 7 to 0 score. In the other four games, B loses by at least 1 goal with a 7 to 6 score. The goal difference is at most $7 \times 3 - 1 \times 4 = 17$. The answer is **(a)**.

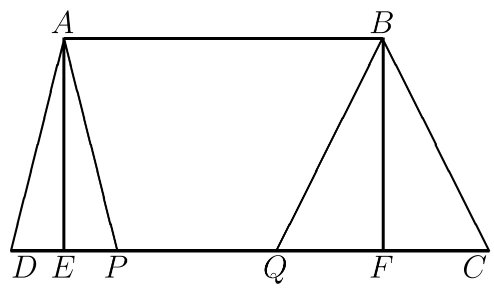
5. $ABCD$ is a quadrilateral with $AB = 12$ and $CD = 18$. Moreover, AB is parallel to CD and both $\angle ADC$ and $\angle BCD$ are less than $90°$. P and Q are points on side CD such that $AD = AP$ and $BC = BQ$. The length of PQ is

(a) 6 (b) 7 (c) 8 (d) 9 (e) 10

Solution:
From A and B, drop perpendiculars onto CD at the points E and F respectively. Since both $\angle ADC$ and $\angle BCD$ are less than $90°$, E and F do lie on the segment CD. Note that

$$DE + FC = CD - EF = CD - AB = 6.$$

Since $ED = EP$ and $FC = FQ$, $EP + FQ = 6 < 12 = EF$, P is closer to E than Q and Q is closer to F than P. Therefore, $PQ = EF - (EP + FQ) = 12 - 6 = 6$. The answer is **(a)**.



6. Each of four cows is either normal or mutant. A normal cow has 4 legs and always lies. A mutant cow has either 3 or 5 legs and always tells the truth. When asked how many legs they have among them, their respective responses are 13, 14, 15 and 16. The total number of legs among these four cows is

(a) 13 (b) 14 (c) 15 (d) 16 (e) none of these

Solution:
Since all four responses are different, at least three of them are wrong. If all four are wrong, then the cows are all normal, and they will have 16 legs among them. However, this makes one of the responses right. Hence one of the responses is indeed right. The three normal cows have 12 legs among them. Hence the mutant cow must have 3 legs in order to make one of the responses right. The answer is **(c)**.

7. Let a and b be positive integers such that $ab < 100$ and $\frac{a}{b} > 2$. Denote the minimum value of $\frac{a}{b}$ by m. Then we have

(a) $m \leq 2.15$ (b) $2.15 < m < 2.2$ (c) $m = 2.2$

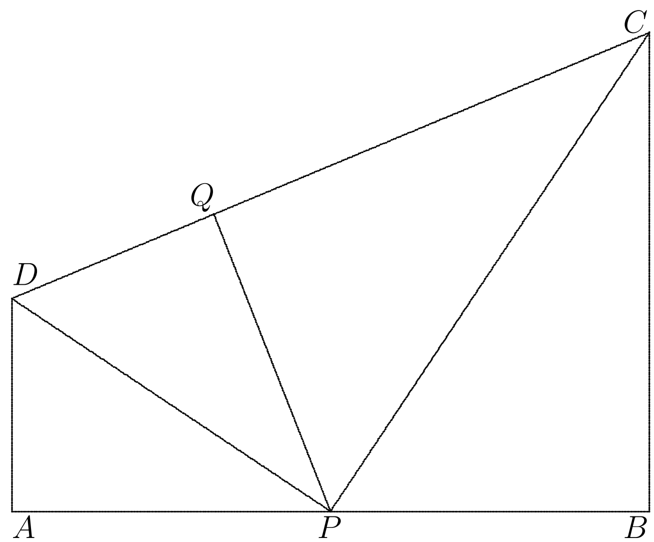
(d) $2.2 < m < 2.25$ (e) $m > 2.25$

Solution:
We have $100 > ab > b(2b+1)$ so that $b \leq 6$. Thus the minimum value of m is $2+\frac{1}{6} = 2.1666\ldots$. The answer is **(b)**.

8. Let $ABCD$ be a quadrilateral with $\angle DAB = \angle CBA = 90°$. Suppose there is a point P on side AB such that $\angle ADP = \angle CDP$ and $\angle BCP = \angle DCP$. If $AD = 8$ and $BC = 18$, the perimeter of the quadrilateral $ABCD$ is

(a) 70 (b) 72 (c) 74 (d) 76 (e) 78

Solution:
Since $\angle DAB + \angle CBA = 180°$, AD is parallel to BC. Therefore, $\angle ADC + \angle BCD = 180°$. Hence $\angle PDC + \angle PCD = 90°$. Consequently, $\angle DPC = 90°$. Let Q be the foot of the perpendicular on CD from P. Note that triangle PDA is congruent to triangle PDQ, and triangle PCB is congruent to triangle PCQ. Hence $DQ = DA = 8$, $CQ = CB = 18$ and $PA = PQ = PB$. Since $\angle DPC = 90°$, triangle DPQ is similar to triangle PCQ. Hence $\frac{DQ}{QP} = \frac{PQ}{QC}$. Therefore, $PQ^2 = 8 \times 18 = 144$ and $PQ = 12$. The perimeter of $ABCD$ is therefore $AP + PB + BC + CQ + QD + DA = 12 + 12 + 18 + 18 + 8 + 8 = 76$. The answer is **(d)**.



9. Two bus routes stop at a certain bus stop. The A bus comes in one-hour intervals while the B bus also comes in regular intervals of a different length. When grandma rests on the bench by the bus stop, one A bus and two B buses come by. Later, grandpa rests on the same bench and eight A buses come by. The *minimum* number of B buses that must have come by during that time is

(a) less than 4 (b) 4 or 5 (c) 6 or 7 (d) 8 or 9 (e) more than 9

Solution:
To minimize the number of B buses coming by, we stretch the length of their intervals as far as possible. Suppose grandma sees the A bus that comes at 10:00. Then she does not see the ones that come at 9:00 or 11:00. Thus the length of the interval between two B buses is strictly less than 2 hours. The two B buses grandma sees may have come at 9:01 and 10:59, in which case the interval is of length 118 minutes. Grandpa is on the bench for at least 7 hours. This is longer than three intervals for the B buses, so he must have seen at least 3 of them. Suppose he sees the A buses at 11:00, 12:00, 13:00, 14:00, 15:00, 16:00, 1700 and 18:00. Then he will see only the B buses at 12:57, 14:55 and 16:53. The answer is **(a)**.

10. Suppose that $16^{2013} = a^b$, where a and b are positive integers. The number of possible values of a is

(a) 2 (b) 8 (c) 11 (d) 16 (e) 24

Solution:
Since $16^{2013} = 2^{4\times 2013}$, a must be of the form 2^k where k is a positive integer divisor of $4 \times 2013 = 2^2 \times 3 \times 11 \times 61$. The prime factorization of k may contain up to two 2s, one 3, one 11 and one 61, so that there are $(2+1)(1+1)^3 = 24$ possible values of k, and therefore of a. The answer is **(e)**.

11. The following five statements are made about the integers a, b, c, d and e: (i) ab is even and c is odd; (ii) bc is even and d is odd; (iii) cd is even and e is odd; (iv) de is even and a is odd; (v) ea is even and b is odd. The maximum number of these statements which may be correct is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution:
If a, c and d are odd while b and e are even, then (i), (ii) and (iv) are all correct. Suppose at least four statements are correct. By symmetry, we may assume that they are (i), (ii), (iii) and (iv). However, by (i) and (ii), c and d are both odd, and yet by (iii), cd is even. This is a contradiction. The answer is **(c)**.

12. A very small cinema has only one row of five seats numbered 1 to 5. Five movie-goers arrive one at a time. Each takes a seat not next to any occupied seat if this is possible. If not, then any seat will do. The number of different orders in which the seats may be taken is

(a) 24 (b) 32 (c) 48 (d) 64 (e) 72

Solution:
The first two movie-goers to arrive may take the pair (1,5), (1,4), (2,5), (1,3), (2,4) or (3,5) of seats. In each case, there are 2!=2 ways for them to do so. If they take (1,5), (1,3) or (3,5), then the third movie-goer has only one choice of seat. The remaining two seats may be occupied in 2!=2 ways. Otherwise, the last three movie-goers may take any vacant seats, and this can be done in 3!=6 ways. Hence the total number of orders in which the seats may be taken is $2 \times 3 \times (2+6) = 48$. The answer is **(c)**.

13. Let $f(x) = x^2 + x + 1$. Let n be the positive number such that $f(n) = f(20)f(21)$. Then the number of distinct prime divisors of n is

(a) 1 (b) 2 (c) 3 (d) 4 (e) more than 4

Solution:
Note that $f(m-1)f(m) = (m^2 - m + 1)(m^2 + m + 1) = m^4 + m^2 + 1 = f(m^2)$. Substituting $m = 21$ yields $f(20)f(21) = f(441)$. Therefore, $n = 441 = 3^2 7^2$. The answer is **(b)**.

14. The number of pairs (x, y) of integers satisfying the equation $x^2 + y^2 + xy - x + y = 2$ is

(a) 3 (b) 4 (c) 5 (d) 6 (e) none of these

Solution:
The equation can be written as $x^2 + (y-1)x + y^2 + y - 2 = 0$. By the Quadratic Formula, the solutions are

$$x = \frac{-(y-1) \pm \sqrt{(y-1)^2 - 4(y^2+y-2)}}{2} = \frac{-(y-1) \pm \sqrt{-3(y-1)(y+3)}}{2}.$$

These are real if and only if $(y+3)(y-1) \leq 0$. Since y is an integer, it must be one of -3, -2, -1,0 and 1. If $y = -3$, then $x = 2$. If $y = -2$, then $x = 0$ or 3. If $y = -1$, x is not an integer. If $y = 0$, then $x = -1$ or 2. Finally, if $y = 1$, then $x = 0$. The answer is **(d)**.

15. A triangle ABC with $AB = 7$, $BC = 8$ and $CA = 10$ has an interior point P such that $\angle APB = \angle BPC = \angle CPA = 120°$. Let r_1, r_2 and r_3 be the radii of the circles passing through the vertices of triangles PAB, PBC and PCA respectively. The value of $r_1^2 + r_2^2 + r_3^2$ is

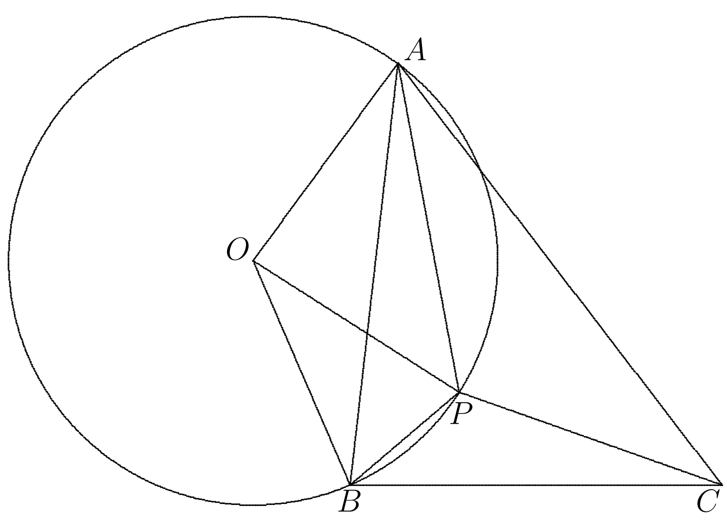
(a) 71 (b) 72 (c) 73 (d) 74 (e) 75

Solution:

Let O be the centre of the circle passing through the vertices of tiangle PAB. Note that since $\angle APB > 90°$, O lies on the perpendicular bisector of AB outside of triangle PAB. Since $OA = OP = PB$,

$$\angle AOB = \angle AOP + \angle POB = 180° - 2\angle APO + 180° - 2\angle BPO = 360° - 2\angle APB = 120°.$$

Hence $AB = \sqrt{3}OA$ so that $OA = \frac{AB}{\sqrt{3}}$. It follows that $r_1^2 = \frac{AB^2}{3}$. Similarly, $r_2^2 = \frac{BC^2}{3}$ and $r_3^2 = \frac{CA^2}{3}$. Hence $r_1^2 + r_2^2 + r_3^2 = \frac{7^2+8^2+10^2}{3} = 71$. The answer is **(a)**.



16. The list 1, 3, 4, 9, 10, 12, 13, ... contains in increasing order all positive integers which can be expressed as sums of one or more distinct integral powers of 3. The 100-th number in this list is

(a) 981 (b) 982 (c) 984 (d) 985 (e) 999

Solution:
If we switch the powers of 3 to the powers to 2, then we get all the positive integers. Hence we convert 100 into base 2, obtaining $100 = 2^6 + 2^5 + 2^2$. It follows that the 100-th number on the list is $3^6 + 3^5 + 3^2 = 981$. The answer is **(a)**.