Large spread does not imply Benford's Law

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Abstract

Sharp universal bounds are given for the distance between normalised Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ and the distribution of $\log X$ mod 1, where X is uniform. The results dispel the popular belief that a random variable obeys Benford's Law (at least approximately) whenever its spread is large.

For every real number x, the largest integer not larger than x will be denoted by $\lfloor x \rfloor$, and $[\![x]\!] := x - \lfloor x \rfloor$ is the fractional (or non-integer) part of x. The base γ logarithm ($\gamma > 1$) of $x > 0$ is $\log_\gamma x$; if used without a subscript, log symbolises the natural logarithm. The sets of natural, non-negative integer, integer, positive real and real numbers are $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}$, respectively.

Given any probability measure μ on $\mathbb{R}$, denote by F_μ its distribution function, that is, $F_\mu(x) = \mu(]-\infty, x])$. For every measurable map $T : \mathbb{R} \to \mathbb{R}$ the probability measure $T\mu$ is defined as $T\mu(B) := \mu\big(T^{-1}(B)\big)$ for all Borel sets B. Specifically, $[\![\mu]\!]$ is, for every μ, concentrated on $[0, 1]$. The uniform distribution on $[a, b]$ with $a < b$ is denoted by $U_{a,b}$. Thus

$$F_{U_{a,b}}(x) = \begin{cases} 0 & \text{if } x < a\,, \\ \dfrac{x-a}{b-a} & \text{if } a \le x < b\,, \\ 1 & \text{if } x \ge b\,. \end{cases}$$

Given any two probability measures μ, ν on $\mathbb{R}$, their Kolmogorov–Smirnov distance $d_\infty(\mu, \nu)$ is

$$d_\infty(\mu, \nu) = \sup_{x \in \mathbb{R}} \big| F_\mu(x) - F_\nu(x) \big|\,;$$

see e.g. [5] for some details on this metric. Recall that μ with $\mu(\mathbb{R}^+) = 1$, or a real random variable with distribution μ, satisfies *Benford's Law* base γ if and only if $\log_\gamma \mu$ is uniform modulo one [2], i.e., if $d_\infty([\![\log_\gamma \mu]\!], U_{0,1})$ equals zero. Contrary to what [4, p.63] may suggest, the uniform distribution $U_{a,b}$ with $a \ge 0$ does not even approximately satisfy Benford's Law for any base γ, no matter how large $b - a$ is.

Theorem 1. *For all $\gamma > 1$ and $0 \le a < b$,*

$$d_\infty\big([\![\log_\gamma U_{a,b}]\!], U_{0,1}\big) \ge C_\gamma > 0\,, \tag{1}$$

where

$$C_\gamma = \frac{1 - \gamma + \log\gamma + (\gamma - 1)\log(\gamma - 1) - (\gamma - 1)\log\log\gamma}{2(\gamma - 1)\log\gamma}\,.$$

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Proof. To verify (1), assume first that $a = 0$, and let $l := \lfloor \log_\gamma b \rfloor$ and $\delta := [\![\log_\gamma b]\!]$. Thus $b = \gamma^{l+\delta}$, and

$$F_{\log_\gamma U_{0,b}}(x) = \begin{cases} \dfrac{\gamma^x}{b} & \text{if } x < l + \delta\,, \\ 1 & \text{if } x \geq l + \delta\,, \end{cases}$$

from which it follows that, for all $0 \leq x \leq 1$,

$$F_{[\![\log_\gamma U_{0,b}]\!]}(x) = \sum\nolimits_{k\in\mathbb{Z}} \bigl(F_{\log_\gamma U_{0,b}}(x+k) - F_{\log_\gamma U_{0,b}}(k)\bigr) = \begin{cases} \dfrac{\gamma^{1+x-\delta} - \gamma^{1-\delta}}{\gamma - 1} & \text{if } 0 \leq x < \delta\,, \\ 1 - \dfrac{\gamma^{1-\delta} - \gamma^{x-\delta}}{\gamma - 1} & \text{if } \delta \leq x \leq 1\,. \end{cases}$$

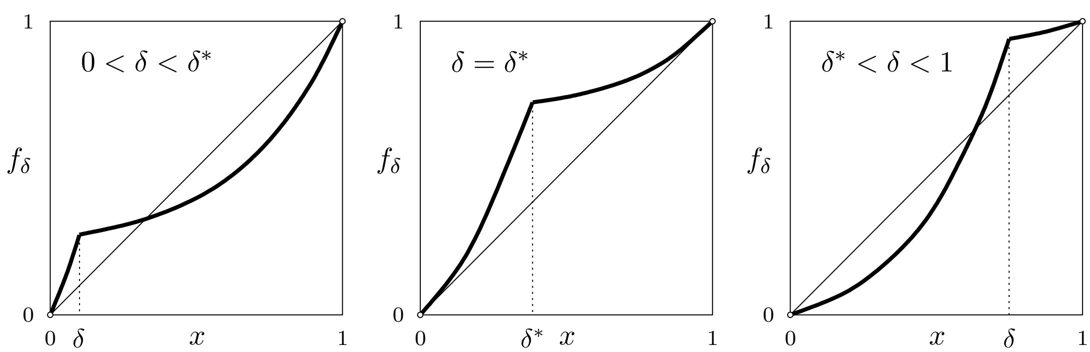
Notice that $F_{[\![\log_\gamma U_{0,b}]\!]}$ does not depend on l. For the sake of brevity, let $f_\delta(x) := F_{[\![\log_\gamma U_{0,b}]\!]}(x)$ and $g_\delta(x) := f_\delta(x) - x$ for all x and δ. Note that f_δ is convex on $[0, \delta]$ and on $[\delta, 1]$, and $f_0(x) = \dfrac{\gamma^x - 1}{\gamma - 1}$, whereas for $0 < \delta < 1$

$$f'_\delta(0+) = f'_\delta(1-) = \frac{\gamma^{1-\delta}\log\gamma}{\gamma - 1}\,.$$

In the latter case, there exists a unique $0 < \delta^* < 1$ such that $f'_{\delta^*}(0+) = f'_{\delta^*}(1-) = 1$; explicitly $\gamma^{1-\delta^*} = (\gamma - 1)/\log\gamma$, and thus

$$\delta^* = \frac{\log\gamma - \log(\gamma - 1) + \log\log\gamma}{\log\gamma}\,.$$

Consequently, for $0 < \delta < 1$ the graph of f_δ can have three qualitatively different forms.



Note that g_δ always attains its maximum at $x = \delta$. This suggests introducing the auxiliary function ψ according to

$$\psi(\delta) := g_\delta(\delta) = f_\delta(\delta) - \delta = \frac{\gamma - \gamma^{1-\delta}}{\gamma - 1} - \delta\,.$$

It follows from

$$\psi'(\delta) = \frac{\gamma^{1-\delta}\log\gamma}{\gamma - 1} - 1 = \gamma^{\delta^* - \delta} - 1\,,$$

that ψ is concave, with $\psi(0) = \psi(1) = 0$ and $\psi'(\delta^*) = 0$. Hence $\psi(\delta) > 0$ for all $0 < \delta < 1$, and

$$\max\nolimits_{0\leq\delta\leq 1}\psi(\delta) = \psi(\delta^*) = \frac{\gamma - \gamma^{1-\delta^*}}{\gamma - 1} - \delta^* = \frac{\gamma\log\gamma - \gamma + 1}{(\gamma - 1)\log\gamma} - \delta^* = 2C_\gamma\,.$$

Assume first that $0 \le \delta \le \delta^*$. In this case, the function g_δ has a non-positive minimum at $x = 1 + \delta - \delta^* > \delta$, with

$$g_\delta(1+\delta-\delta^*) = 1 - \frac{\gamma^{1-\delta} - \gamma^{1-\delta^*}}{\gamma - 1} - 1 - \delta + \delta^* = \psi(\delta) - \psi(\delta^*)\,,$$

showing that $\max_{0\le x\le 1} g_\delta(x) = \psi(\delta)$ as well as $-\min_{0\le x\le 1} g_\delta(x) = \psi(\delta^*) - \psi(\delta)$. Similarly, if $\delta^* < \delta < 1$ then g_δ has a negative minimum at $x = \delta - \delta^* < \delta$, with

$$g_\delta(\delta - \delta^*) = \frac{\gamma^{1-\delta^*} - \gamma^{1-\delta}}{\gamma - 1} - \delta + \delta^* = \psi(\delta) - \psi(\delta^*)\,.$$

For all $0 \le \delta < 1$, therefore,

$$\max_{0\le x\le 1} g_\delta(x) = \psi(\delta)\,, \quad -\min_{0\le x\le 1} g_\delta(x) = \psi(\delta^*) - \psi(\delta)\,,$$

and consequently

$$\max_{0\le x\le 1} |g_\delta(x)| = \max\bigl\{\psi(\delta), \psi(\delta^*) - \psi(\delta)\bigr\} \ge \tfrac{1}{2}\psi(\delta^*) = C_\gamma\,,$$

which establishes (1) for the case $a = 0$.

To verify (1) for $a > 0$ assume for the time being that $\log_\gamma a = k \in \mathbb{Z}$, and let $l := \lfloor \log_\gamma b \rfloor$ and $\delta := [\![\log_\gamma b]\!]$ as before; for convenience set $m := l - k \in \mathbb{N}_0$. A short computation confirms that

$$f_{m,\delta}(x) := F_{[\![\log_\gamma U_{a,b}]\!]}(x) = \begin{cases} \dfrac{\gamma^x - 1}{\gamma - 1} \cdot \dfrac{\gamma^{m+1} - 1}{\gamma^{m+\delta} - 1} & \text{if } 0 \le x < \delta\,, \\[2ex] 1 - \dfrac{\gamma - \gamma^x}{\gamma - 1} \cdot \dfrac{\gamma^m - 1}{\gamma^{m+\delta} - 1} & \text{if } \delta \le x \le 1\,. \end{cases}$$

Notice that $f_{m,\delta} \to f_\delta$ uniformly on $[0,1]$ as $m \to \infty$. Let again $g_{m,\delta}(x) := f_{m,\delta}(x) - x$ and observe that

$$g_{m,\delta}(x) - g_\delta(x) = f_{m,\delta}(x) - f_\delta(x) = \Delta_{m,\delta}(x)\,,$$

where $\Delta_{m,\delta}$ is given by

$$\Delta_{m,\delta}(x) = \begin{cases} \dfrac{\gamma^x - 1}{\gamma - 1} \cdot \dfrac{\gamma^{1-\delta} - 1}{\gamma^{m+\delta} - 1} & \text{if } 0 \le x < \delta\,, \\[2ex] \dfrac{\gamma^{1-\delta} - \gamma^{x-\delta}}{\gamma - 1} \cdot \dfrac{\gamma^\delta - 1}{\gamma^{m+\delta} - 1} & \text{if } \delta \le x \le 1\,. \end{cases}$$

Obviously, $\Delta_{m,\delta} \ge 0$ with $\Delta_{m,\delta}(0) = \Delta_{m,\delta}(1) = 0$, and $\Delta_{m,0} = 0$ for all $m \ge 1$. Furthermore, for $0 < \delta < 1$ the function $\Delta_{m,\delta}$ is convex and increasing on $[0,\delta]$, and concave and decreasing on $[\delta, 1]$. Since both g_δ and $\Delta_{m,\delta}$ attain their respective maximal value at $x = \delta$,

$$\max_{0\le x\le 1} g_{m,\delta}(x) = g_{m,\delta}(\delta) = g_\delta(\delta) + \Delta_{m,\delta}(\delta) = \psi(\delta) + \Delta_{m,\delta}(\delta)\,.$$

If $0 \le \delta \le \delta^*$ then, with the appropriate $0 \le \xi \le 1$,

$$\begin{aligned} \max_{0\le x\le 1} g_{m,\delta}(x) - \min_{0\le x\le 1} g_{m,\delta}(x) &= g_{m,\delta}(\delta) - g_{m,\delta}(\xi) \\ &\ge g_{m,\delta}(\delta) - g_{m,\delta}(1+\delta-\delta^*) \\ &= \psi(\delta) + \Delta_{m,\delta}(\delta) - g_\delta(1+\delta-\delta^*) - \Delta_{m,\delta}(1+\delta-\delta^*) \\ &= \psi(\delta^*) + \Delta_{m,\delta}(\delta) - \Delta_{m,\delta}(1+\delta-\delta^*) \\ &\ge \psi(\delta^*) \\ &= 2C_\gamma\,. \end{aligned}$$

The same argument applies for $\delta^* < \delta < 1$ with $1 + \delta - \delta^*$ replaced by $\delta - \delta^*$. Thus

$$\max_{0 \le x \le 1} g_{m,\delta}(x) - \min_{0 \le x \le 1} g_{m,\delta}(x) \ge 2C_\gamma \tag{2}$$

holds for all $m \in \mathbb{N}_0$ and $0 \le \delta < 1$, and this in turn implies (1) since

$$\max_{0 \le x \le 1} |g_{m,\delta}(x)| \ge \tfrac{1}{2}\big(\max_{0 \le x \le 1} g_{m,\delta}(x) - \min_{0 \le x \le 1} g_{m,\delta}(x)\big) \ge C_\gamma \, .$$

Overall, therefore, the proof is complete if $a = 0$ or $\log_\gamma a \in \mathbb{Z}$.

Finally, assume that $a > 0$ does not satisfy $\log_\gamma a \in \mathbb{Z}$, that is, the number $\tau := [\![\log_\gamma a]\!]$ lies strictly between 0 and 1. Note that

$$[\![\log_\gamma U_{a,b}]\!] = [\![\log_\gamma U_{a\gamma^{-\tau}, b\gamma^{-\tau}} + \tau]\!] \, ,$$

and clearly $\log_\gamma(a\gamma^{-\tau}) \in \mathbb{Z}$. It is readily verified that, for every non-atomic probability measure μ on $\mathbb{R}$ and every $t \in \mathbb{R}$,

$$F_{[\![\mu+t]\!]}(x) = \begin{cases} F_{[\![\mu]\!]}(x + 1 - [\![t]\!]) - F_{[\![\mu]\!]}(1 - [\![t]\!]) & \text{if } 0 \le x < [\![t]\!] \, , \\ F_{[\![\mu]\!]}(x - [\![t]\!]) + 1 - F_{[\![\mu]\!]}(1 - [\![t]\!]) & \text{if } [\![t]\!] \le x \le 1 \, , \end{cases}$$

and therefore also

$$G_{[\![\mu+t]\!]}(x) = \begin{cases} G_{[\![\mu]\!]}(x + 1 - [\![t]\!]) - G_{[\![\mu]\!]}(1 - [\![t]\!]) & \text{if } 0 \le x < [\![t]\!] \, , \\ G_{[\![\mu]\!]}(x - [\![t]\!]) - G_{[\![\mu]\!]}(1 - [\![t]\!]) & \text{if } [\![t]\!] \le x \le 1 \, , \end{cases}$$

where generally $G_\mu(x) := F_\mu(x) - x$. In particular,

$$\max_{0 \le x \le 1} G_{[\![\mu+t]\!]}(x) - \min_{0 \le x \le 1} G_{[\![\mu+t]\!]}(x) = \max_{0 \le x \le 1} G_{[\![\mu]\!]}(x) - \min_{0 \le x \le 1} G_{[\![\mu]\!]}(x) \, , \tag{3}$$

which merely expresses the intuitively obvious fact that the *span* (i.e. the difference between maximal and minimal value) of $G_{[\![\mu]\!]}$ is not affected by the rotation caused by adding (modulo one) any number t. With the notation introduced earlier, $G_{[\![\log_\gamma U_{a\gamma^{-\tau}, b\gamma^{-\tau}}]\!]} = g_{m,\delta}$, where $m = \lfloor \log_\gamma b/a \rfloor$ and $\delta = [\![\log_\gamma b/a]\!]$. Combining (2) and (3) for $\mu = \log_\gamma U_{a\gamma^{-\tau}, b\gamma^{-\tau}}$ and $t = \tau$ therefore yields

$$\max_{0 \le x \le 1} G_{[\![\log_\gamma U_{a,b}]\!]}(x) - \min_{0 \le x \le 1} G_{[\![\log_\gamma U_{a,b}]\!]}(x) \ge 2C_\gamma \, .$$

This completes the proof. □

Remark 2. (i) As the above argument shows, the constant C_γ in (1) is best possible: For every $C > C_\gamma$ there exist a, b with $0 < a < b$ such that $d_\infty([\![\log_\gamma U_{a,b}]\!], U_{0,1}) < C$.

(ii) It follows from the first part of the proof of Theorem 1 that, for all $\gamma > 1$ and $b > 0$,

$$d_\infty\big([\![\log_\gamma U_{0,b}]\!], U_{0,1}\big) = \Psi(\log_\gamma b) \, ,$$

with the continuous, 1-periodic function $\Psi : x \mapsto \max\big\{\psi([\![x]\!]), 2C_\gamma - \psi([\![x]\!])\big\} = C_\gamma + |\psi([\![x]\!]) - C_\gamma|$.

(iii) Note that $\gamma \mapsto C_\gamma$ is monotonically increasing, with $\lim_{\gamma \to 1+} C_\gamma = 0$ and $\lim_{\gamma \to \infty} C_\gamma = \frac{1}{2}$. For $\gamma = 10$, the most important special case in view of Benford's Law, one finds $C_{10} \approx 0.13442$.

(iv) Satisfactory though it may be, Theorem 1 has a small shortcoming: For every probability measure μ on $\mathbb{R}$, the measure $[\![\mu]\!]$ naturally lives on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ rather than on $[0, 1]$, but d_∞ is unsuitable for measures on $\mathbb{T}$. Specifically, $\mathbb{T}$ is a compact metric space when endowed with the metric $d(x + \mathbb{Z}, y + \mathbb{Z}) := \min_{k \in \mathbb{Z}} |x - y + k|$, and consequently $\mathcal{P}(\mathbb{T})$, the space of all probability measures on $\mathbb{T}$ with the topology of weak convergence, is compact and metrizable [3]. A natural metric inducing this topology is the Kantorovich–Wasserstein distance d_K defined as

$$d_K(\mu, \nu) := \sup\left\{ \left| \int_{\mathbb{T}} f \, \mathrm{d}\mu - \int_{\mathbb{T}} f \, \mathrm{d}\nu \right| \, : \, f \in C_{\mathbb{R}}(\mathbb{T}), \mathrm{Lip} f \le 1 \right\} \, .$$

Unlike d_∞, the metric d_K on $\mathcal{P}(\mathbb{T})$ is invariant under isometries of $\mathbb{T}$: If $T : \mathbb{T} \to \mathbb{T}$ is any isometry, then $d_K(T\mu, T\nu) = d_K(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}(\mathbb{T})$. Explicit practicable formulae for d_K have been derived in [1]. A truly satisfactory variant of Theorem 1, therefore, would consider $[\![\log_\gamma U_{a,b}]\!]$ an element of $\mathcal{P}(\mathbb{T})$ and provide a lower bound for its distance from $\lambda_\mathbb{T}$, the uniform distribution on $\mathbb{T}$. Such a result can indeed be achieved using parts of the proof of Theorem 1 even though the necessary calculations are significantly more involved. The final result, however, is even slightly simpler than (1): For all $\gamma > 1$ and $0 \le a < b$,

$$d_K\big([\![\log_\gamma U_{a,b}]\!], \lambda_\mathbb{T}\big) \ge \log_\gamma \tfrac{1+\sqrt{\gamma}}{2} - \tfrac{1}{4} =: \tfrac{1}{4}\Phi(\tfrac{1}{4}\log\gamma) > 0\,, \tag{4}$$

where Φ is the real-analytic odd function $\Phi(x) = x^{-1}\log\cosh x$; as in the case of (1), the inequality (4) is best possible in the sense of (i).

References

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