Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 68 (2008) 3789–3811
Nonlinear Analysis
www.elsevier.com/locate/na

Uniformly attracting solutions of nonautonomous differential equations

A. Berger[a,1], S. Siegmund[b,*]

[a] *Department of Mathematics and Statistics, University of Canterbury, Christchurch, New Zealand*
[b] *Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, Frankfurt am Main, Germany*

Received 5 February 2007; accepted 19 April 2007

Abstract

Understanding the structure of attractors is fundamental in nonautonomous stability and bifurcation theory. By means of clarifying theorems and carefully designed examples we highlight the potential complexity of attractors for nonautonomous differential equations that are as close to autonomous equations as possible. We introduce and study bounded uniform attractors and repellors for nonautonomous scalar differential equations, in particular for asymptotically autonomous, polynomial, and periodic equations. Our results suggest that uniformly attracting or repelling solutions are the true analogues of attracting or repelling fixed points of autonomous systems. We provide sharp conditions for the autonomous structure to break up and give way to a bewildering diversity of nonautonomous bifurcations.

MSC: primary 34A12; 34D45; 37C70; secondary 34C07; 34C11; 37B55

Keywords: Nonautonomous dynamical system; Attractor; Repellor; Asymptotically autonomous; Polynomial differential equation; Poincaré map

1. Introduction

From the plethora of tools for analysing the asymptotic behaviour of autonomous differential equations

$$\dot{x} = F(x), \tag{1}$$

many have been extended to nonautonomous equations

$$\dot{x} = f(t, x), \tag*{$(1)_t$}$$

e.g. spectral theory [38,42], invariant manifold theory [3,48], or the Hartman–Grobman theorem [4,34,43]. As an important common feature, all these generalisations assume uniformity in time $t \in \mathbb{R}$ of some sort or other, for instance via exponential dichotomies of a linearisation or via Lipschitz conditions independent of t. Imposing uniformity

* Corresponding author. Tel.: +49 69 798 22517; fax: +49 69 798 28846.
E-mail addresses: arno.berger@canterbury.ac.nz (A. Berger), siegmund@math.uni-frankfurt.de (S. Siegmund).
[1] Tel.: +64 3 364 2682; fax: +64 3 364 2587.

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.04.020

conditions is reasonable if a priori knowledge on the time dependence of f is available, e.g. if f is periodic, almost periodic or automorphic in t [41], but it may become questionable if the dependence on t is more complex. For a comprehensive study of bifurcation scenarios for instance an analysis of more general time dependencies is indispensable.

The recent past has seen numerous attempts to extend bifurcation theory to nonautonomous differential equations [9,13,16–19,22,25,27,28,36]. While classical bifurcation theory for dynamical systems (1) describes the change of stability and the creation, annihilation and break up of equilibria, periodic, heteroclinic and homoclinic orbits etc., it is, in the nonautonomous situation of $(1)_t$, not at all obvious the bifurcations of which objects one should study; equilibria and periodic orbits for instance are not generic for $(1)_t$ if f depends aperiodically on t. One natural approach is to study the transition of *attractors* [25]. For example, a classical (supercritical) two-dimensional Hopf bifurcation could be understood as a transition from a singleton attractor (consisting of an asymptotically stable equilibrium) to a more complex attractor (bounded by an attractive periodic orbit and containing in its interior a repelling equilibrium). While a classical attractor for (1) is simply a compact invariant set which attracts a neighbourhood, there exist several non-equivalent definitions for nonautonomous attractors, e.g. forward and pullback attractors [6] describing, respectively, the future and the past of $(1)_t$. Consequently, in order to better understand nonautonomous bifurcation scenarios one has to analyse the structure and transition of these attractors whenever Eq. $(1)_t$ depends on a parameter. Typically, attractors depend merely upper semi-continuously on parameters; the dependence is continuous only if the attraction is uniform w.r.t. the parameter [29]. Clearly, establishing the existence of some (globally) attracting object can only be the first step of an in-depth analysis. Looking *inside* an attractor, however, usually is much more demanding. To the best of our knowledge no systematic study of internal attractor bifurcations for nonautonomous equations exists so far. Bifurcation results available often rely on the fact that attracting solutions can be computed and analysed explicitly [25,27] or employ somewhat technical or restrictive assumptions on the system under consideration, as for instance in the formidable balance of Taylor coefficients utilised in [28].

The purpose of this article is to lay the foundations for internal attractor bifurcation studies. In the simplest possible setting we illustrate by means of clarifying theorems and carefully designed examples the potential structural complexity of attractors for equations which are as close to autonomous differential equations as possible. We study attraction and repulsion properties of individual solutions (i.e., the natural analogues of autonomous equilibria) of nonautonomous scalar differential equation $(1)_t$. In Section 2 we propose the definition of a *bounded uniform attractor* which incorporates two of the most popular mechanisms of attraction, namely *forward* and *pullback* attraction. In the autonomous situation of (1), all notions of attraction coincide. In general, a uniform attractor for $(1)_t$ is substantially more than just a forward attractor that is also pullback attracting (see Examples 5 and 12); as far as we know, this important point has not been clarified in the literature before. In Section 3 we study in detail the properties of uniform attractors and repellors for nonautonomous equation $(1)_t$ where we maintain various autonomous features by focusing on equations that are, respectively, *asymptotically autonomous* (Section 3.1), *polynomial* in x (Section 3.2), and *periodic* in t (Section 3.3). While all notions of attraction coincide for scalar periodic equations (Theorem 23), in general the classes of forward, pullback, and uniform attractors are all different, the latter being strictly smaller than the intersection of the two former.

From Section 3.2 onward, we pay special attention to uniform attractors and repellors of Eq. $(1)_t$ that are polynomial in x,

$$\dot{x} = \varepsilon \left(a_0(t) + a_1(t)x + \cdots + a_d(t)x^d \right) \tag{2}$$

with $\varepsilon > 0$. In the autonomous case, that is, for $a_0, a_1, \ldots, a_d$ not depending on t, the total number of uniformly attracting and repelling solutions of (2) is, quite trivially, bounded by d. The latter bound turns out to be correct as well if either (2) is asymptotically autonomous, if $\varepsilon > 0$ is sufficiently small, or if $d \leq 3$, where in the case $d = 3$ additional assumptions have to be imposed on a_3 (see Theorem 21 for details). Quite different arguments are required to deal with each situation. Moreover, none of the assumptions mentioned can be omitted. We demonstrate this by means of a series of results which unite and profoundly extend ideas scattered throughout the literature. To give but one example, even for periodic coefficients a_i Eq. (2) may have any (finite) number of uniform attractors if $d \geq 3$ — provided that ε is not too small (Theorem 28). Thus a non-trivial bifurcation problem emerges for (2) upon variation of ε. Our results strongly suggest that the notion of uniform attractors and repellors will be useful in tackling problems of this kind. Accordingly, we conclude this article by indicating several fundamental questions that arise naturally from the results presented.

2. Bounded uniform attractors and repellors

In this article we consider nonautonomous scalar differential equations

$$\dot{x} = f(t, x), \tag{3}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ as well as $\frac{\partial}{\partial x} f$ is continuous in (t, x). Moreover, we assume throughout that $\sup_{\mathbb{R}\times K} |f| < \infty$ for every compact set $K \subset \mathbb{R}$. Given $(t_0, x_0) \in \mathbb{R}^2$ the initial value problem consisting of (3) together with $x(t_0) = x_0$ has a unique solution $t \mapsto \varphi(t; t_0, x_0)$ defined on some (possibly bounded) maximal open interval containing t_0. Although solutions of (3) may (and often will) become unbounded in finite time we are particularly interested in *bounded* solutions. A bounded solution $\mu : \mathbb{R} \to \mathbb{R}$ of (3) can attract or repel neighbouring solutions in different ways.

Example 1. Let ψ denote the bounded C^1 function

$$\psi : t \mapsto \frac{t|t|}{1+t^2} \quad (t \in \mathbb{R}),$$

which, together with its primitive

$$\Psi : t \mapsto \int_0^t \psi(s)\mathrm{d}s = |t| - |\arctan t| \quad (t \in \mathbb{R}),$$

will be used several times throughout this article as an auxiliary function. With this, consider the equation

$$\dot{x} = \cos t + \psi(t)\sin t - \psi(t)x, \tag{4}$$

which, for all $(t_0, x_0) \in \mathbb{R}^2$, has the solution

$$\varphi(t; t_0, x_0) = \mathrm{e}^{\Psi(t_0)-\Psi(t)}(x_0 - \sin t_0) + \sin t.$$

Obviously, all solutions of (4) exist for all time, every solution is bounded, and

$$\lim_{t\to\infty} |\varphi(t; t_0, x_0) - \sin t| = 0, \quad \forall (t_0, x_0) \in \mathbb{R}^2. \tag{5}$$

Thus the bounded solution $\mu_1 : t \mapsto \sin t$ of (4) attracts all solutions in forward time (see Fig. 1). It is an example of a (global) *forward attractor*.

Example 2. Analogously, for the equation

$$\dot{x} = \cos t - \psi(t)\sin t + \psi(t)x, \tag{6}$$

we find, for all $(t_0, x_0) \in \mathbb{R}^2$,

$$\varphi(t; t_0, x_0) = \mathrm{e}^{\Psi(t)-\Psi(t_0)}(x_0 - \sin t_0) + \sin t.$$

Now $\mu_2 : t \mapsto \sin t$ is the only bounded solution of (6). Evidently, it is not attracting in the sense of the previous example (see Fig. 1). However, μ_2 attracts all solutions in the sense that, at any given time t, solutions satisfying an initial condition at time t_0 much smaller than t are close to $\mu_2(t)$. Formally

$$\lim_{t_0\to-\infty} |\varphi(t; t_0, x_0) - \sin t| = 0, \quad \forall t \in \mathbb{R}, x_0 \in \mathbb{R}. \tag{7}$$

Thus μ_2 is an example of a (global) *pullback attractor*.

It is obvious that μ_1 is not pullback attracting for (4) whereas μ_2 is not forward attracting for (6). Evidently, this discrepancy is due to the highly non-uniform convergence in (5) and (7). Observations like these naturally motivate parts (iii) and (iv) of Definition 3. This definition may look slightly unfamiliar to the reader proficient in the theory of nonautonomous attractors, as developed e.g. in [7,8,23,24]. The subsequent Remark 4 explicates why Definition 3 nevertheless is the most appropriate for our purposes.

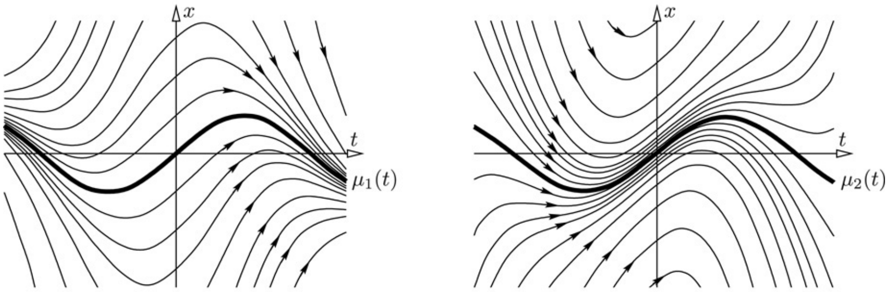

Fig. 1. Forward (left, see Example 1) and pullback attraction (Example 2) are different concepts.

Definition 3. Let $\mu : \mathbb{R} \to \mathbb{R}$ be a bounded solution of (3) and $(x_{0,\sigma})_{\sigma\in\mathbb{R}}$ a family of real numbers. The solution μ is called

(i) a *forward attractor* if there exists $\delta > 0$ such that, for every $t_0 \in \mathbb{R}$,

$$\lim_{\tau\to\infty} |\varphi(t_0 + \tau; t_0, x_{0,t_0}) - \mu(t_0 + \tau)| = 0,$$

whenever $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$;

(ii) a *pullback attractor* if there exists $\delta > 0$ such that, for every $t \in \mathbb{R}$,

$$\lim_{\sigma\to-\infty} |\varphi(t; \sigma, x_{0,\sigma}) - \mu(t)| = 0,$$

whenever $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$;

(iii) a *uniform attractor* if there exists $\delta > 0$ such that

$$\lim_{\tau\to\infty} \|\varphi(\cdot + \tau; \cdot, x_{0,\cdot}) - \mu(\cdot + \tau)\|_\infty = \lim_{\tau\to\infty} \sup_{\sigma\in\mathbb{R}} |\varphi(\sigma + \tau; \sigma, x_{0,\sigma}) - \mu(\sigma + \tau)| = 0,$$

provided that $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$;

(iv) a *uniform repellor* if there exists $\delta > 0$ such that

$$\lim_{\tau\to\infty} \|\varphi(\cdot - \tau; \cdot, x_{0,\cdot}) - \mu(\cdot - \tau)\|_\infty = \lim_{\tau\to\infty} \sup_{\sigma\in\mathbb{R}} |\varphi(\sigma - \tau; \sigma, x_{0,\sigma}) - \mu(\sigma - \tau)| = 0,$$

provided that $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$.

We call μ a *global* forward, pullback, uniform attractor or uniform repellor if the respective property above holds for *every* $\delta > 0$.

Remark 4. (i) Forward and pullback attractors are usually defined much more generally as families (parametrised by t) of non-empty compact sets which are attractive and invariant in some sense or other. Attraction typically occurs only within an *attracting universe* [23], that is, a parametrised family of non-empty bounded sets satisfying further consistency and regularity conditions. Since we are interested in attraction phenomena on the finest scale we focus on attractors and repellors which are *solutions* of (3), i.e., every t-fibre is a singleton. Thus Definition 3 really is a tailor-made specialisation of more general concepts. Also note that the solutions of (3) are ordered, that is, $\varphi(t; t_0, x_0) < \varphi(t; t_0, y_0)$ for all t whenever $x_0 < y_0$. Consequently, the condition $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$ in Definition 3 corresponds to an attractive universe consisting of all nonautonomous sets $(D_\sigma)_{\sigma\in\mathbb{R}}$ with $\sup_{x\in D_\sigma} |x - \mu(\sigma)| < \delta$ for all $\sigma \in \mathbb{R}$ or, equivalently, the attracting universe generated by the set-valued map $\sigma \mapsto]\mu(\sigma) - \delta, \mu(\sigma) + \delta[$. For a global attractor or repellor μ, therefore, the corresponding universe would comprise, at any $\sigma \in \mathbb{R}$, all bounded sets containing $\mu(\sigma)$.

(ii) In principle, one could allow the solution μ in Definition 3 to be unbounded. If so, (3) may have more attractors than is desirable. For a trivial example take $f(t, x) \equiv -x$ in (3); *every* solution is then a uniform attractor, but only the equilibrium $\mu = 0$ is bounded. Also, from a practical point of view (cf. [26]), it may be hard to distinguish actual divergence from convergence towards an unbounded "attractor".

(iii) The notions of uniform attractor and repellor exhibit a natural duality. Such natural dual counterparts exist neither for forward nor for pullback attractors [6,21,25,27].

Example 5. Trivially, every uniform attractor is both a forward and a pullback attractor. The converse is not true in general as the following example shows.

For every $a \in \mathbb{R}$ let h_a denote the bounded continuous function

$$h_a : t \mapsto 1 - a - \pi \max\left(0, \sin\sqrt{|t|}\right) \quad (t \in \mathbb{R}),$$

and consider the equation

$$\dot{x} = \cos t - h_a(t)\sin t + h_a(t)x, \tag{8}$$

for which a short computation yields

$$\varphi(t; t_0, x_0) = (x_0 - \sin t_0)e^{\int_{t_0}^t h_a(s)\mathrm{d}s} + \sin t \quad \forall (t_0, x_0) \in \mathbb{R}^2. \tag{9}$$

Since h_a is even, the solution $\mu : t \mapsto \sin t$ of (8) is a forward attractor if and only if it is a pullback attractor, which in turn is the case precisely if $a > 0$. On the other hand, given any $\tau > 0$, pick $N \in \mathbb{N}$ such that $(2N+1)^2\pi^2 + \tau < (2N+2)^2\pi^2$, that is, $N > \frac{1}{4}\pi^{-2}(\tau - 3\pi^2)$. Choosing in particular $t_0 = (2N+1)^2\pi^2$ we immediately deduce from (9) that

$$\sup_{\sigma\in\mathbb{R}} |\varphi(\sigma + \tau; \sigma, x_{0,\sigma}) - \sin(\sigma + \tau)| \geq |x_0 - \sin t_0|e^{\tau(1-a)}.$$

Thus μ is not a uniform attractor if $a \leq 1$. For $a > 1$ obviously $h_a \leq 1 - a < 0$, so

$$\sup_{\sigma\in\mathbb{R}} |\varphi(\sigma + \tau; \sigma, x_{0,\sigma}) - \sin(\sigma + \tau)| \leq e^{\tau(1-a)} \sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \sin\sigma| \to 0 \quad \text{as } \tau \to \infty,$$

which shows that μ is a global uniform attractor if and only if $a > 1$.

Example 6. Two bounded solutions μ_1, μ_2 of (3) may be separated in the sense that their distance $\inf_{t\in\mathbb{R}} |\mu_1(t) - \mu_2(t)|$ is positive. Any uniform attractor is necessarily separated from any uniform repellor (cf. the proof of Theorem 19). It is, however, worth noting that $\lim_{t\to\infty} |\mu_1(t) - \mu_2(t)| = 0$ for two uniform attractors μ_1, μ_2 is not ruled out by Definition 3. Concretely, consider the equation

$$\dot{x} = \frac{1}{2}(1 - \psi(t))x - \frac{1}{2}x^3, \tag{10}$$

the solutions of which are given by

$$\varphi(t; t_0, x_0) = \frac{x_0}{\sqrt{e^{\Psi(t)-t-\Psi(t_0)+t_0} + x_0^2\int_{t_0}^t e^{\Psi(t)-t-\Psi(s)+s}\mathrm{d}s}} \quad \forall (t_0, x_0) \in \mathbb{R}^2,\ t \geq t_0. \tag{11}$$

Taking the limit $t_0 \to -\infty$ in (11) yields two candidates for pullback attractors for (10), namely μ and $-\mu$, where

$$\mu : t \mapsto \frac{1}{\sqrt{\int_{-\infty}^t e^{\Psi(t)-t-\Psi(s)+s}\mathrm{d}s}} \quad (t \in \mathbb{R}).$$

It is easy to check that μ and $-\mu$ are indeed bounded monotone solutions of (10) and that $\lim_{t\to\infty}\mu(t) = 0$ (see Fig. 2). The introduction of a new coordinate ξ via $x = \xi\mu$ transforms (10) into

$$\dot{\xi} = \frac{1}{2}\mu(t)^2\xi(1 - \xi^2),$$

from which an equivalent form of (11),

$$\varphi(t; t_0, x_0) = \mu(t)\frac{x_0 e^{\frac{1}{2}\int_{t_0}^t \mu(s)^2\mathrm{d}s}}{\sqrt{\mu(t_0)^2 - x_0^2 + x_0^2 e^{\int_{t_0}^t \mu(s)^2\mathrm{d}s}}} \quad \forall (t_0, x_0) \in \mathbb{R}^2,\ t \geq t_0, \tag{12}$$

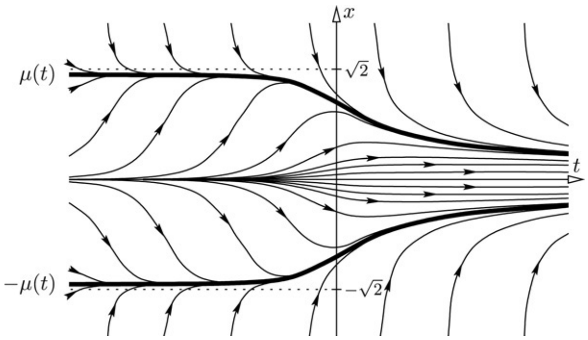

Fig. 2. The two uniform attractors μ and $-\mu$ of (10) are not separated.

can be derived easily. The representation (12) is particularly useful for showing that μ and $-\mu$ are in fact uniform attractors of (10). Since $\lim_{t\to\infty} |\mu(t) - (-\mu(t))| = 0$ these two attractors are evidently not separated.

From Theorem 13 it follows that $\lim_{t\to-\infty} \mu(t) = \sqrt{2}$. This theorem also implies that the zero solution of (10) which in Fig. 2 appears somewhat repelling is not a uniform repellor; while it is not a uniform attractor either, it is obviously a forward attractor.

Example 7. To observe an infinite number of attractors that are not separated consider

$$\dot{x} = -\mathrm{e}^t(1+\mathrm{e}^t)^{-1}x + (1+\mathrm{e}^t)^{-1}H\left(x(1+\mathrm{e}^t)\right), \tag{13}$$

where H denotes the C^∞ function

$$H(x) = \begin{cases} 0 & \text{if } x = 0, \\ \mathrm{e}^{-|x|^{-1}}\sin(x^{-1}) & \text{if } x \neq 0. \end{cases}$$

For every $k \in \mathbb{Z} \setminus \{0\}$ the function

$$\mu_k : t \mapsto \frac{1}{k\pi}(1+\mathrm{e}^t)^{-1} \quad (t \in \mathbb{R}),$$

is a solution of (13). As in Example 6 it is straightforward to check that μ_k is a uniform attractor whenever k is even. Note also that for odd k the solution μ_k, albeit somewhat repelling, is not a uniform repellor but – like *every* solution of (13) – a forward attractor.

The following simple fact implicitly underlies most of the subsequent arguments, and we recall it here for the reader's convenience, see e.g. [47].

Proposition 8. *Let $\mu_1 < \mu_2$ be two bounded solutions of (3). Then the solution of (3) with $x(t_0) = x_0$ exists for all $t \in \mathbb{R}$ whenever $\mu_1(t_0) \le x_0 \le \mu_2(t_0)$.*

3. Properties of uniform attractors

A uniform attractor resembles an attracting equilibrium in the autonomous case. We now analyse analogies and differences between the two concepts. Recall that throughout we assume f in (3), and also $\frac{\partial}{\partial x} f$, to be continuous in (t, x) and bounded on every stripe $\mathbb{R} \times K$, K compact.

3.1. Autonomous and asymptotically autonomous differential equations

In the autonomous case, that is for $f(t, x) \equiv F(x)$ not depending on t, all notions of attraction coincide, and attractors are easily characterised in terms of F. Essentially, this is due to the invariance of (1) under time translations.

Theorem 9. *Let F be C^1 and $\mu : \mathbb{R} \to \mathbb{R}$ a bounded solution of $\dot{x} = F(x)$. Then the following statements are equivalent:*

(i) *μ is a forward attractor;*
(ii) *μ is a pullback attractor;*
(iii) *μ is a uniform attractor;*
(iv) *μ is constant, $\mu(t) \equiv \mu_0$, and μ_0 is an isolated zero of F such that $F(a) > 0 > F(b)$ for all $a < \mu_0 < b$ with $b - a$ sufficiently small.*

Proof. Current as well as later arguments become especially transparent by means of α- and ω-limit sets and their nonautonomous counterparts [15,33]. Quite generally, therefore, for every bounded continuous function $\nu : \mathbb{R} \to \mathbb{R}$ we denote by $\alpha(\nu)$ and $\omega(\nu)$ the set of all backward and forward accumulation points of $\{\nu(t) : t \in \mathbb{R}\}$, respectively, i.e.,

$$\alpha(\nu) = \{x \in \mathbb{R} : \exists (t_n) \text{ s.t. } t_n \to \infty, \nu(-t_n) \to x\},$$
$$\omega(\nu) = \{x \in \mathbb{R} : \exists (t_n) \text{ s.t. } t_n \to \infty, \nu(t_n) \to x\}.$$

It is well known (and easy to see) that $\alpha(\nu)$, $\omega(\nu)$ are non-empty compact intervals (which may, of course, degenerate to single points).

Every solution μ of $\dot{x} = F(x)$ is monotonic, and therefore the limits $\lim_{t\to-\infty} \mu(t) = \mu_-$ and $\lim_{t\to\infty} \mu(t) = \mu_+$ exist if μ is bounded. Thus $\alpha(\mu) = \{\mu_-\}$ and $\omega(\mu) = \{\mu_+\}$. Clearly, $F(\mu_-) = F(\mu_+) = 0$. Assume now that μ is a forward attractor. If $\mu_- \neq \mu_+$ and $\delta > 0$, then there exists t_0 such that $|\mu_- - \mu(t_0)| < \delta$ yet

$$|\varphi(t_0 + \tau; t_0, \mu_-) - \mu(t_0 + \tau)| = |\mu_- - \mu(t_0 + \tau)| \to |\mu_- - \mu_+| \neq 0 \quad \text{as } \tau \to \infty,$$

contradicting the fact that μ is a forward attractor. Hence $\mu_+ = \mu_- = \mu_0$, and μ is constant. Similarly, if μ is a pullback attractor then, for all $\sigma \leq t_0$ with t_0 sufficiently small, $|\mu_- - \mu(\sigma)| < \delta$ and so, choosing $x_{0,\sigma}$ equal to μ_- or $\mu(\sigma)$, depending on whether $\sigma \leq t_0$ or $\sigma > t_0$, we find

$$\lim_{\sigma\to-\infty} |\varphi(t; \sigma, x_{0,\sigma}) - \mu(t)| = |\mu_- - \mu(t)| \neq 0,$$

unless μ is constant. In either case therefore $\mu(t) \equiv \mu_0$ and $F(\mu_0) = 0$. If there existed a strictly increasing sequence (a_n) with $a_n \to \mu_0$ and $F(a_n) \leq 0$ for all n then $\varphi(t; t_0, a_n)$ would neither converge to μ_0 as $t \to \infty$ nor as $t_0 \to -\infty$. A completely analogous argument yields that $F(b) < 0$ for all b larger than but sufficiently close to μ_0. Overall, this shows that (i) and (ii), and hence also (iii), imply (iv).

Conversely, assume that $F(\mu_0) = 0$ and $F(a) > 0 > F(b)$ for all a, b with $a_0 \leq a < \mu_0 < b \leq b_0$ and appropriate $a_0 < \mu_0 < b_0$. For every $x_0 \in [a_0, b_0]$ the solution $\varphi(t; t_0, x_0)$ exists for all $t \geq t_0$, and, with $\delta = \min(b_0 - \mu_0, \mu_0 - a_0) > 0$,

$$\begin{aligned}\sup_{\sigma\in\mathbb{R}} |\varphi(\sigma + \tau; \sigma, x_{0,\sigma}) - \mu_0| &= \sup_{\sigma\in\mathbb{R}} |\varphi(\tau; 0, x_{0,\sigma}) - \mu_0| \\ &\leq \max(|\varphi(\tau; 0, a_0) - \mu_0|, |\varphi(\tau; 0, b_0) - \mu_0|) \to 0 \quad \text{as } \tau \to \infty,\end{aligned}$$

provided that $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$. Thus $\mu = \mu_0$ is a uniform attractor. □

Corollary 10. *Let F be C^1 and $\mu : \mathbb{R} \to \mathbb{R}$ a bounded solution of $\dot{x} = F(x)$. Then μ is a uniform repellor if and only if $\mu(t) \equiv \mu_0$ and μ_0 is an isolated zero of F such that $F(a) < 0 < F(b)$ for all $a < \mu_0 < b$ with $b - a$ sufficiently small.*

Proof. We only have to observe that μ is a uniform repellor if and only if $t \mapsto \mu(-t)$ is a uniform attractor for the equation $\dot{x} = -F(x)$. □

Recall that (3) is termed *asymptotically autonomous* if for two (necessarily continuous) functions $f_-, f_+ : \mathbb{R} \to \mathbb{R}$

$$\lim_{t\to-\infty} f(t, x) = f_-(x), \qquad \lim_{t\to\infty} f(t, x) = f_+(x),$$

holds locally uniformly in x. (See e.g. [33,44,45] for information and references on the extensive theory of asymptotically autonomous systems; note that solutions of the limiting equations $\dot{x} = f_-(x)$ and $\dot{x} = f_+(x)$ may fail to be unique if f_-, f_+ are not Lipschitz continuous.) In some respects, asymptotically autonomous systems resemble autonomous ones, but there are also various counterexamples to the intuition that the asymptotic behaviour of (3) and its limit equations $\dot{x} = f_-(x)$, $\dot{x} = f_+(x)$ are necessarily the same [45]. The latter point is emphasised by the following example which shows that the equivalence of all notions of attraction, as established by Theorem 9 for the autonomous case, does not even extend to asymptotically autonomous systems.

Example 11. Let a_0, a_1 denote the bounded C^1 functions

$$a_0(t) = \mathrm{e}^{-\Psi(t)} \quad \text{and} \quad a_1(t) = -1 - \psi(t) \quad (t \in \mathbb{R}),$$

respectively, with the functions ψ and Ψ introduced in Example 1, and consider the asymptotically autonomous equation

$$\dot{x} = a_0(t) + a_1(t)x, \tag{14}$$

for which $f_-(x) \equiv 0$ and $f_+(x) = -2x$. An elementary computation confirms that

$$\mu : t \mapsto \mathrm{e}^{-\Psi(t)} = a_0(t) \quad (t \in \mathbb{R}),$$

is a global forward but not a pullback attractor for (14).

Similarly, choosing C^1 functions a_0, a_1 as

$$a_0(t) = \mathrm{e}^{\Psi(t)-t^2}(1-2t) \quad \text{and} \quad a_1(t) = -1 + \psi(t) \quad (t \in \mathbb{R}),$$

yields another asymptotically autonomous equation (14) with $f_-(x) = -2x$, $f_+(x) \equiv 0$. In this case, the bounded solution

$$\mu : t \mapsto \mathrm{e}^{\Psi(t)-t^2} \quad (t \in \mathbb{R})$$

fails to be forward attracting but nevertheless is a global pullback attractor.

Example 12. In the setting of the previous example, let

$$a_0(t) = \mathrm{e}^{-t^2}(1-2t), \qquad a_1(t) = |t|^{-\frac{1}{4}} \min\left(0, \sin\sqrt{|t|}\right) \quad (t \in \mathbb{R}).$$

Again, (14) is asymptotically autonomous with $f_+(x) = f_-(x) \equiv 0$. Define $A_1(t) = \int_0^t a_1(s)\mathrm{d}s$, $t \in \mathbb{R}$. Then the C^1 function

$$\mu : t \mapsto \int_{-\infty}^{t} a_0(s)\mathrm{e}^{A_1(t)-A_1(s)}\mathrm{d}s \quad (t \in \mathbb{R}),$$

is a bounded solution of (14) with $\lim_{|t|\to\infty} \mu(t) = 0$. It is easy to check that μ is both a global forward and pullback attractor. An argument similar to the one in Example 5 shows that μ is not a *uniform* attractor, essentially because A_1 contains arbitrarily long constant sections.

Thus for asymptotically autonomous equations all concepts discussed so far are different. Nevertheless, for uniform attractors Theorem 9 in a sense extends to such equations.

Theorem 13. *Assume (3) is asymptotically autonomous, and let $\mu : \mathbb{R} \to \mathbb{R}$ be a uniform attractor. Then the limits*

$$\lim_{t\to-\infty} \mu(t) = \mu_-, \qquad \lim_{t\to\infty} \mu(t) = \mu_+,$$

exist and satisfy $f_-(\mu_-) = f_+(\mu_+) = 0$. Moreover, for all $a < \mu_+ < b$ with $b - a$ sufficiently small, $f_+(a) \geq 0 \geq f_+(b)$. Similarly, $f_-(c) \geq 0 \geq f_-(d)$ for all $c < \mu_- < d$ with $d - c$ sufficiently small. Furthermore, f_- and f_+ do not vanish identically on any neighbourhood of μ_- and μ_+, respectively.

Proof. We only prove the assertions concerning f_+ and μ_+ because the arguments for backward time are completely analogous. Assume that μ is a uniform attractor and let $\omega(\mu) = [\omega_1, \omega_2]$. Since $\liminf_{t\to\infty} \mu(t)$ would be smaller or larger than ω_1 if $f_+(\omega_1)$ were, respectively, negative or positive, we must have $f_+(\omega_1) = 0$, and in fact $f_+(x) = 0$ for every $x \in \omega(\mu)$.

Assume now that $\omega_1 < \omega_2$ and define $a(t) = \sup_{x\in\omega(\mu)} |f(t,x)|$, $t \in \mathbb{R}$, as well as $c = \frac{2}{3}\omega_1 + \frac{1}{3}\omega_2$, $d = \frac{1}{3}\omega_1 + \frac{2}{3}\omega_2$. We can find a strictly increasing sequence (s_n) with $s_n \to \infty$ such that, for all $n \in \mathbb{N}$, $\mu([s_n, s_n + n]) \subset [c, d]$ and $\sup_{t\geq s_n} a(t) < \frac{1}{6}(\omega_2 - \omega_1)/n$, but also $\int_{s_n}^{s_n+n} a(s)\mathrm{d}s \to 0$ as $n \to \infty$. Given $\delta > 0$, let $\delta' = \min\left(\delta, \frac{1}{6}(\omega_2 - \omega_1)\right)$

and assume that $|x_0 - \mu(s_n)| < \delta'$. Then both $\mu(t)$ and $\varphi(t; s_n, x_0)$ are contained in $\omega(\mu)$ for all $s_n \le t \le s_n + n$. Assuming without loss of generality that $\int_{s_n}^{s_n+n} a(s)\mathrm{d}s < \frac{1}{8}\delta'$ for all n and that $\mu(s_n) > x_0$, we deduce from the trivial estimate

$$\mu(t) - \varphi(t; s_n, x_0) \ge \mu(s_n) - x_0 - 2\int_{s_n}^{t} a(s)\mathrm{d}s \quad \forall t \in [s_n, s_n + n],$$

that $\mu(t) - \varphi(t; s_n, x_0) \ge |\mu(s_n) - x_0| - \frac{1}{4}\delta'$ for all $t \in [s_n, s_n + n]$. Let $x_{0,\sigma} = \mu(s_n) - \frac{1}{2}\delta'$ if $\sigma = s_n$ for some n, and $x_{0,\sigma} = \mu(\sigma)$ otherwise. Then $\mu(t) - \varphi(t; s_n, x_{0,s_n}) \ge \frac{1}{4}\delta'$ for all $t \in [s_n, s_n + n]$ and therefore

$$\limsup_{\tau\to\infty} \sup_{\sigma\in\mathbb{R}} |\varphi(\sigma + \tau; \sigma, x_{0,\sigma}) - \mu(\sigma + \tau)| \ge \frac{1}{4}\delta',$$

while $\|x_{0,\cdot} - \mu(\cdot)\|_\infty < \delta'$. This contradicts the fact that μ is a uniform attractor. Consequently, $\omega_1 = \omega_2 = \mu_+$.

The preceding arguments also show that f_+ cannot vanish identically on any neighbourhood of μ_+. Finally, if $f_+(a_n) < 0$ for all $n \in \mathbb{N}$ and some increasing sequence (a_n) with $a_n \to \mu_+$ then, for t_0 sufficiently large, $\varphi(t; t_0, a_n)$ does not converge to μ_+ as $t \to \infty$, which clearly contradicts the fact that μ is an attractor. □

Corollary 14. *If μ is a uniform repellor of the asymptotically autonomous equation $\dot{x} = f(t, x)$ then the limits $\lim_{t\to-\infty} \mu(t) = \mu_-$, $\lim_{t\to\infty} \mu(t) = \mu_+$ exist and satisfy $f_-(\mu_-) = f_+(\mu_+) = 0$. For all $a < \mu_+ < b$ with $b - a$ sufficiently small, $f_+(a) \le 0 \le f_+(b)$, and for all $c < \mu_- < d$ with $d - c$ sufficiently small, $f_-(c) \le 0 \le f_-(d)$. Also, f_- and f_+ do not vanish identically on any neighbourhood of μ_- and μ_+, respectively.*

Example 15. Theorem 13 and Corollary 14 should be seen as extensions of, respectively, Theorem 9 and Corollary 10 to asymptotically autonomous equations and their uniform attractors. An analogous statement for forward or pullback attractors is false. For example, the equation

$$\dot{x} = \frac{4\sqrt{2}t^2|t|}{1 + t^4}\cos\left(\log(1 + t^4)\right) - \frac{4t^2|t|}{1 + t^4}x$$

is asymptotically autonomous, with $f_+ = f_- = 0$, and has

$$\mu : t \mapsto \sin\left(\frac{\pi}{4} + \operatorname{sign} t \cdot \log(1 + t^4)\right) \quad (t \in \mathbb{R})$$

as a global forward and pullback attractor. (Here, as usual, $\operatorname{sign} t$ equals $1, 0$ or -1, depending on whether $t \in \mathbb{R}$ is positive, zero or negative.) It follows from Theorem 13 that μ is not a *uniform* attractor; this can also be confirmed by a short direct computation.

Example 16. Unlike Theorem 9 for the autonomous case, Theorem 13 does not assert that μ_+ is an *isolated* zero of f_+. In general, this cannot be expected as the following example shows.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(t, x) = \begin{cases} 0 & \text{if } x = 0, \\ -x\mathrm{e}^{-|x|^{-1}}\left((1 + \mathrm{e}^{-t^2}\operatorname{sign} x)\sin^2(x^{-1}) + \mathrm{e}^{-t^2}\right)^{\frac{1}{4}} & \text{if } x \ne 0, \end{cases}$$

so that f is C^∞ in t and x, and $f(t, x) \to f_\pm(x)$ locally uniformly as $|t| \to \infty$, where

$$f_\pm(x) = \begin{cases} 0 & \text{if } x = 0, \\ -x\mathrm{e}^{-|x|^{-1}}|\sin(x^{-1})|^{\frac{1}{2}} & \text{if } x \ne 0. \end{cases}$$

Obviously, $\mu = 0$ is a solution of $\dot{x} = f(t, x)$. As before let $t \mapsto \varphi(t; t_0, x_0)$ denote the (unique) solution of the latter equation with $x(t_0) = x_0$. Furthermore, let w be *any* solution of the autonomous initial value problem

$$\dot{w} = f_\pm(w), \quad w(0) = x_0. \tag{15}$$

As $(f_+(x) - f(t, x))\operatorname{sign} x \ge 0$ for all $(t, x) \in \mathbb{R}^2$ it follows from elementary differential inequality considerations [47, p. 95] that $(w(t - t_0) - \varphi(t; t_0, x_0))\operatorname{sign} x_0 \ge 0$ for all $t \ge t_0$. Choosing in particular w as the minimal solution of (15) if $x_0 > 0$, and as the maximal solution if $x_0 < 0$, we see that μ is indeed a uniform attractor, but clearly $\mu_+ = 0$ is not an isolated zero of $f_\pm$.

Example 16 is somewhat pathological in that (15) does not have a unique solution. An additional regularity assumption on the limiting functions simplifies the situation.

Theorem 17. *If, under the assumptions of Theorem 13, the functions f_- and f_+ are C^1 then μ_- and μ_+ are isolated zeros of f_- and f_+, respectively.*

Proof. Again it is enough to verify the assertion for forward time. Assume that $f_+(\mu_n) = 0$ for all $n \in \mathbb{N}$ and some monotone sequence (μ_n) with $\mu_n \to \mu_+$; without loss of generality let (μ_n) be strictly decreasing. We are going to show that this assumption is incompatible with μ being a uniform attractor — provided that f_+ is C^1.

If $f'_+(\mu_n) \neq 0$ for infinitely many n then there exist points $x > \mu_+$ with $x - \mu_+ > 0$ arbitrarily small such that $f(t, x)$ is positive for all sufficiently large t. This, however, is impossible because μ is an attractor. Without loss of generality we can thus assume that $f'_+(\mu_n) = 0$ for all $n \in \mathbb{N}$. Given $\delta > 0$, choose $N \in \mathbb{N}$ such that $|\mu(t) - \mu_N| < \delta$ for all $t \geq N$. For every $k \in \mathbb{N}$ there exists $0 < \varepsilon_k < \frac{1}{3}(\mu_N - \mu_+)$ such that $|f'_+(x)| < \frac{1}{2k}$ whenever $|x - \mu_N| < \varepsilon_k$. Consequently, we can find a strictly increasing sequence $(T_k)_{k\in\mathbb{N}}$ with $T_1 > N$ such that $|f(t, x)| < \frac{\varepsilon_k}{k}$ for all $t \geq T_k$ and all $x \in [\mu_N - \varepsilon_k, \mu_N + \varepsilon_k]$, and also $\mu(t) < \frac{2}{3}\mu_+ + \frac{1}{3}\mu_N$ whenever $t \geq T_1$. It follows that $\varphi(t; T_k, \mu_N)$ is an element of $[\mu_N - \varepsilon_k, \mu_N + \varepsilon_k]$ for (at least) all $t \in [T_k, T_k + k]$. Choose now $x_{0,\sigma} = \mu_N$ if $\sigma = T_k$ for some k, and $x_{0,\sigma} = \mu(\sigma)$ otherwise. Clearly, $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$. Observing that

$$\begin{aligned}\sup_{\sigma\in\mathbb{R}} |\varphi(\sigma + \tau; \sigma, x_{0,\sigma}) - \mu(\sigma + \tau)| &\geq |\varphi(T_k + \tau; T_k, \mu_N) - \mu(T_k + \tau)| \\ &\geq |\mu_N - \mu(T_k + \tau)| - |\mu_N - \varphi(T_k + \tau; T_k, \mu_N)| \\ &> \frac{2}{3}(\mu_N - \mu_+) - \frac{1}{3}(\mu_N - \mu_+) = \frac{1}{3}(\mu_N - \mu_+) > 0,\end{aligned}$$

for all $\tau \in [0, k]$, we deduce that $\|\varphi(\cdot + \tau; \cdot, x_{0,\cdot}) - \mu(\cdot + \tau)\|_\infty \not\to 0$ as $\tau \to \infty$ and that therefore μ cannot be a uniform attractor. This contradiction completes the proof. □

Remark 18. (i) Theorem 17 applies in particular if $\frac{\partial}{\partial x} f$ converges locally uniformly as $t \to -\infty$ and $t \to \infty$, respectively. In this case, and unlike in Example 16, long-term regularity prevails over spatial complexity (as measured by $\frac{\partial}{\partial x} f$). In one way or the other, most questions regarding uniform attractors touch upon a subtle balance between temporal and spatial complexity.

(ii) The function $f_\pm$ in Example 16 is not Lipschitz continuous. Since it guarantees the uniqueness of solutions of (15), Lipschitz continuity of f_- and f_+ is often incorporated in the definition of asymptotically autonomous differential equations [33,44]. One may wonder whether Theorem 17 remains valid with f_- and f_+ being merely Lipschitz continuous. No proof or counterexample is yet known to the authors.

3.2. Polynomial differential equations

From now on we focus on Eq. (3) with f polynomial in x, that is, on equations

$$\dot{x} = a_0(t) + a_1(t)x + \cdots + a_d(t)x^d, \tag{16}$$

where $d \in \mathbb{N}_0$ does not depend on t and $a_0, a_1, \ldots, a_d$ are bounded continuous functions. Motivated by the obvious bound in the autonomous case, that is for $a_0, a_1, \ldots, a_d$ not depending on t, we shall be interested in the maximal number of uniform attractors and repellors that (16) can have. Generally, the problem of counting special solutions of (16), notably periodic ones (see Section 3.3), has attracted considerable interest, not least for its connection with Hilbert's Sixteenth Problem, and has consequently been discussed under many different perspectives, see e.g. [1,10,20,30–32,35,40,46] and the many references therein.

For asymptotically autonomous equation (16) a bound on the numbers of uniform attractors and repellors follows immediately from Theorem 13 and Corollary 14.

Theorem 19. *If the limits $\lim_{t\to-\infty} a_i(t)$, $\lim_{t\to\infty} a_i(t)$ exist for every $i = 0, 1, \ldots, d$ then the total number of uniform attractors and repellors of (16) is at most d.*

Proof. We shall use the following informal terminology: Two bounded solutions μ_1, μ_2 of (3) are *joined at* $-\infty$ and ∞ if, respectively, $\lim_{t\to-\infty} |\mu_1(t) - \mu_2(t)| = 0$ and $\lim_{t\to\infty} |\mu_1(t) - \mu_2(t)| = 0$. Two attractors can be joined at ∞ (recall Example 6). It is, however, impossible for two uniform attractors μ_1, μ_2 to be joined at $-\infty$ (and thus also for two uniform repellors to be joined at ∞). Indeed, if $\mu_1 < \mu_2$ were joined at $-\infty$ then, for every $\delta > 0$ there would exist $t_0 < 0$ such that $0 < \mu_2(t) - \mu_1(t) < \delta$ for all $t \le t_0$; with

$$x_{0,\sigma} = \begin{cases} \mu_2(\sigma) & \text{if } \sigma \le t_0, \\ \mu_1(\sigma) & \text{if } \sigma > t_0, \end{cases}$$

therefore $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu_1(\sigma)| < \delta$. However, for all $\tau > -t_0$

$$\|\varphi(\cdot + \tau; \cdot, x_{0,\cdot}) - \mu_1(\cdot + \tau)\|_\infty \ge |\varphi(0; -\tau, x_{0,-\tau}) - \mu_1(0)| = \mu_2(0) - \mu_1(0) > 0,$$

so $\|\varphi(\cdot + \tau; \cdot, x_{0,\cdot}) - \mu_1(\cdot + \tau)\|_\infty \not\to 0$ as $\tau \to 0$, which clearly contradicts the fact that μ_1 is a uniform attractor. Completely similar reasoning shows that no uniform attractor can be joined at all to any uniform repellor.

To prove the theorem, let $\mu_-^{(1)} < \cdots < \mu_-^{(d_-)}$ and $\mu_+^{(1)} < \cdots < \mu_+^{(d_+)}$ denote the different zeros of f_- and f_+, respectively, where clearly $d_-, d_+ \le d$. (We can assume that neither f_- nor f_+ vanishes identically as otherwise (16) would not have any uniform attractors or repellors.) Also, let $\mu_1 < \cdots < \mu_L$ be a finite family of uniform attractors and repellors of (16). According to Theorem 13 there exist numbers $1 \le k_1 \le \cdots \le k_L \le d_-$ and $1 \le l_1 \le \cdots \le l_L \le d_+$ such that

$$\lim_{t\to-\infty} \mu_i = \mu_-^{(k_i)}, \qquad \lim_{t\to\infty} \mu_i = \mu_+^{(l_i)} \quad (i = 1, \ldots, L).$$

If μ_i is a uniform attractor then $l_{i+1} \ge l_i$ because solutions do not intersect over time. If $l_{i+1} = l_i$ then μ_{i+1} must also be an attractor, and $k_{i+1} \ge k_i + 2$. This follows from the fact that for every solution x between μ_i and μ_{i+1} the set $\alpha(x)$ must contain a zero of f_- between $\mu_-^{(k_i)}$ and $\mu_-^{(k_{i+1})}$ that is different from both numbers. If, on the other hand, $l_{i+1} > l_i$ then μ_{i+1} could be an attractor or a repellor and thus $k_{i+1} \ge k_i + 1$. In either case therefore $k_{i+1} + l_{i+1} \ge k_i + l_i + 2$. Completely analogous reasoning shows that the latter inequality also holds if μ_i is a uniform repellor. Since clearly $k_1 + l_1 \ge 2$ we have $k_i + l_i \ge 2i$ for all i. On the other hand, $k_i + l_i \le d_- + d_+$ for all i so that $L \le \frac{1}{2}(d_- + d_+)$. Eq. (16) can thus have at most $\lfloor \frac{1}{2}(d_- + d_+) \rfloor \le d$ uniform attractors and repellors. (Here, as usual, $\lfloor t \rfloor$ denotes the largest integer not larger than $t \in \mathbb{R}$.) $\square$

Remark 20. (i) The bound $\lfloor \frac{1}{2}(d_- + d_+) \rfloor$ provided by the proof of Theorem 19 is sharp, as can be seen for instance from Eq. (10) in Example 6 which exhibits two uniform attractors and for which $d_- = 3$, $d_+ = 1$.

(ii) It should be noted that the stipulated uniformity and the polynomial structure of (16) are both indispensable in Theorem 19. Example 7 for instance presents an asymptotically autonomous $f = f(t, x)$ for which $f_- = H$ is not polynomial in x and for which (3) has infinitely many uniform attractors. On the other hand, with $d = 1$ and

$$a_0(t) \equiv 0, \qquad a_1(t) = -\psi(t) \quad (t \in \mathbb{R}),$$

every solution of (16) is bounded and a forward attractor, while clearly none is a *uniform* attractor.

Without the assumption of asymptotic autonomy the situation becomes more intricate. We first extend Theorem 19 to equations for which d is small.

Theorem 21. *Let $d = 0, 1$ or 2 and $a_0, \ldots, a_d$ be bounded continuous functions. Then the total number of uniform attractors and repellors of (16) is at most d. This is true also for $d = 3$, provided that a_3 is either non-negative or non-positive and $\int_{-\infty}^0 |a_3(t)| \mathrm{d}t = \int_0^\infty |a_3(t)| \mathrm{d}t = \infty$.*

Proof. We deal with each case separately. If $d = 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_2 - x_1) = 0$$

for any two solutions x_1, x_2 of (16). Hence $x_2 - x_1$ is constant and no solution can attract or repel any other.

If $d = 1$ then, for any three different solutions $x_1 < x_2 < x_3$ of (16),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_2 - x_1}{x_3 - x_1}\right) = \frac{x_2 - x_1}{x_3 - x_1}\left(\frac{\dot{x}_2 - \dot{x}_1}{x_2 - x_1} - \frac{\dot{x}_3 - \dot{x}_1}{x_3 - x_1}\right) = \frac{x_2 - x_1}{x_3 - x_1}(a_1 - a_1) = 0,$$

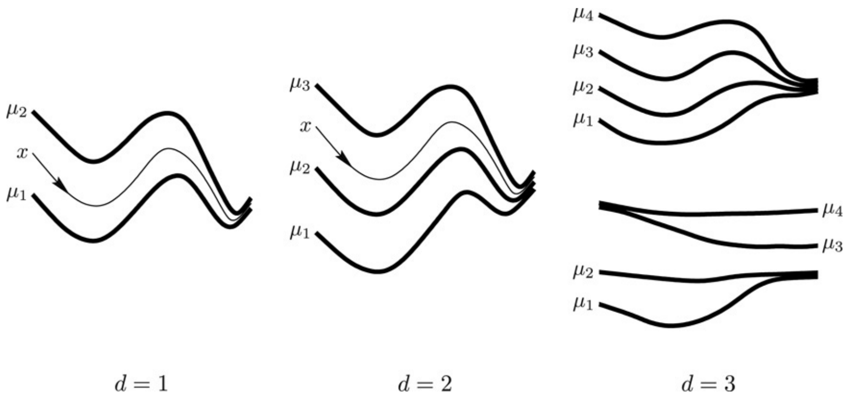

Fig. 3. Ruling out the existence of $d + 1$ uniform attractors and repellors for (16) with $d = 1, 2, 3$.

so $\frac{x_2 - x_1}{x_3 - x_1}$ is constant. Assume that (16) has two uniform attractors $\mu_1 < \mu_2$ that are joined at ∞. Then, for every solution x between μ_1 and μ_2, $x = \mu_1 + \rho(\mu_2 - \mu_1)$ with some constant $0 < \rho < 1$. Clearly this contradicts the uniformity of μ_1 and μ_2 and therefore no two attractors (and similarly no two repellors) of (16) can be joined for $d = 1$ (see left part of Fig. 3). If, however, $\mu_1 < \mu_2$ are two bounded solutions with $\inf_{t \in \mathbb{R}} |\mu_1(t) - \mu_2(t)| > 0$ then neither of them can be attracting or repelling. Overall, we conclude that (16) has at most *one* uniform attractor or repellor if $d = 1$.

If $d \leq 3$ then, for any four different solutions $x_1 < x_2 < x_3 < x_4$ of (16),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_3 - x_1}{x_2 - x_1}\frac{x_4 - x_2}{x_4 - x_3}\right) = \frac{x_3 - x_1}{x_2 - x_1}\frac{x_4 - x_2}{x_4 - x_3}(-a_3)(x_4 - x_1)(x_3 - x_2). \tag{17}$$

For $d = 2$, therefore, the cross ratio $CR(x_1, x_2, x_3, x_4) = \frac{x_3 - x_1}{x_2 - x_1}\frac{x_4 - x_2}{x_4 - x_3}$ is constant. As before, assume that three uniform attractors $\mu_1 < \mu_2 < \mu_3$ are joined at ∞. Then, for every solution x between μ_2 and μ_3,

$$x = (1 - \rho)\mu_1 + \rho\mu_3 \quad \text{with } \rho = \frac{\gamma(\mu_2 - \mu_1)}{\mu_3 - \mu_2 + \gamma(\mu_2 - \mu_1)},$$

where γ is a positive constant. Note that ρ now depends on t but, as before, $0 < \rho(t) < 1$ for all $t \in \mathbb{R}$. From $\mu_3 - x = (1 - \rho)(\mu_3 - \mu_1)$ we see that μ_3 cannot be a uniform attractor (see middle part of Fig. 3). Thus at most two uniform attractors (or repellors) can be joined if $d = 2$. If, however, $\inf_{t \in \mathbb{R}} |\mu_3(t) - \mu_2(t)| > 0$ then $\sup_{t \in \mathbb{R}} \rho(t) < 1$ and μ_3 cannot possibly be uniformly attracting in either forward or backward time. Therefore, (16) has at most *two* uniform attractors and repellors for $d = 2$.

Finally, consider the case $d = 3$ and let $\mu_1 < \mu_2 < \mu_3 < \mu_4$ be four bounded solutions of (16). Assume first that $m = \min_{i=1}^{3} \inf_{t \geq 0} |\mu_{i+1}(t) - \mu_i(t)| > 0$, that is, no two solutions are joined at ∞, and let $M = \max_{i=1}^{3} \sup_{t \geq 0} |\mu_{i+1}(t) - \mu_i(t)|$. If $a_3 \geq 0$ then we deduce from (17), for any $t_0 \geq 0$,

$$CR(\mu_1, \mu_2, \mu_3, \mu_4) \leq CR(\mu_1, \mu_2, \mu_3, \mu_4)|_{t_0} \mathrm{e}^{-m^2 \int_{t_0}^{t} a_3(s)\mathrm{d}s}.$$

Thus $CR\,(\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)) \to 0$ as $t \to \infty$, contradicting the obvious (lower) bound

$$CR(\mu_1, \mu_2, \mu_3, \mu_4) \geq \frac{m^2}{M^2} > 0 \quad \forall t \geq 0.$$

On the other hand, for $a_3 \leq 0$, (17) implies

$$CR(\mu_1, \mu_2, \mu_3, \mu_4) \geq CR(\mu_1, \mu_2, \mu_3, \mu_4)|_{t_0} \mathrm{e}^{-m^2 \int_{t_0}^{t} a_3(s)\mathrm{d}s},$$

so $CR\,(\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)) \to \infty$ as $t \to \infty$, which clearly contradicts the upper bound

$$CR(\mu_1, \mu_2, \mu_3, \mu_4) \leq \frac{M^2}{m^2} \quad \forall t \geq 0.$$

The only conceivable configuration containing four uniform attractors and repellors that is not covered by the preceding argument and its "mirrored" version (i.e., considering $t \le 0$ instead of $t \ge 0$) consists of a pair of attractors ($\mu_1 < \mu_2$, say) joined at ∞ and a pair of repellors ($\mu_3 < \mu_4$, say, with $\mu_2 < \mu_3$) joined at $-\infty$ (see lower right part of Fig. 3; the configuration depicted in the upper right part of Fig. 3 for instance is ruled out by the above argument for $t \to -\infty$.). In this case, rewrite (17) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t} CR(\mu_1, \mu_3, \mu_2, \mu_4) = CR(\mu_1, \mu_3, \mu_2, \mu_4) a_3(\mu_4 - \mu_1)(\mu_3 - \mu_2),$$

to deduce that $t \mapsto CR(\mu_1, \mu_3, \mu_2, \mu_4)$ is monotone. For the configuration considered here, that is, for the case $\lim_{t\to\infty} (\mu_2(t) - \mu_1(t)) = \lim_{t\to-\infty} (\mu_4(t) - \mu_3(t)) = 0$, we have

$$\lim_{|t|\to\infty} CR(\mu_1, \mu_3, \mu_2, \mu_4) = 0.$$

Consequently $CR(\mu_1, \mu_3, \mu_2, \mu_4) = 0$, an obvious absurdity. With this it has been demonstrated that (17) cannot have four uniform attractors and repellors if $d = 3$. □

To the unprejudiced reader Theorem 21 must seem unsatisfactory, its statement being dictated solely by the limitations of the simple arguments used in the course of the proof. It may thus come as a surprise that Theorem 21 is in a sense best possible, as will become apparent in the next section (Theorem 28 and Corollary 29).

3.3. Periodic differential equations

Apart from the asymptotically autonomous case, the simplest situation occurs in (3) if f is T-periodic in the first argument, that is, if

$$f(t + T, x) = f(t, x), \quad \forall (t, x) \in \mathbb{R}^2, \tag{18}$$

with some $T > 0$. We shall see shortly that, as far as uniform attractors and repellors are concerned, even this apparently tame class of equations allows for considerable structural complexity. First, however, we recall a few simple general facts about periodic differential equations. For the latter, the complete information about all solutions of (3) is encoded in the family of local homeomorphisms $\Phi_s : x \mapsto \varphi(s; 0, x)$, $0 \le s \le T$, and in particular in Φ_T, the *Poincaré map* associated with (3). More formally, for all $(t_0, x_0) \in \mathbb{R}^2$ and $t \in \mathbb{R}$, and whenever $\varphi(t; t_0, x_0)$ is defined,

$$\varphi(t; t_0, x_0) = \Phi_{s_2} \circ \Phi_T^k \circ \Phi_{s_1}^{-1}(x_0),$$

where $s_1 = t_0 - T\lfloor t_0/T \rfloor$, $s_2 = t - T\lfloor t/T \rfloor$, and $k = \lfloor t/T \rfloor - \lfloor t_0/T \rfloor$, hence $0 \le s_1, s_2 < T$ and $k \in \mathbb{Z}$. Therefore the long-term dynamics of (3) is governed by Φ_T. For instance, μ is a pT-periodic solution with $p \in \mathbb{N}$ if and only if $\mu(0)$ is a fixed point of Φ_T^p. Also, it is easy to see that the period of every non-constant periodic solution of (3) is a rational multiple of T, and in fact has to be of the form $\frac{1}{q} T$ for some $q \in \mathbb{N}$ (i.e., the solution is *superharmonic*) because Φ_T is order-preserving [14,41].

Lemma 22. *Let f satisfy (18) and assume that $\mu : \mathbb{R} \to \mathbb{R}$ is a (forward, pullback, or uniform) attractor or a uniform repellor of (3). Then μ is T-periodic.*

Proof. Let μ be a bounded solution and observe that, since Φ_T is monotone and $\mu(kT) = \Phi_T^k(\mu(0))$ for all $k \in \mathbb{Z}$, the sequence $(\mu(kT))_{k\in\mathbb{Z}}$ is monotone and bounded. Hence the limits $\mu_- = \lim_{k\to-\infty} \mu(kT)$ and $\mu_+ = \lim_{k\to\infty} \mu(kT)$ exist; moreover, $\Phi_T(\mu_-) = \mu_-$ and $\Phi_T(\mu_+) = \mu_+$. Let ν_-, ν_+ denote the T-periodic solutions corresponding to these two fixed points of Φ_T, that is,

$$\nu_-(t) = \varphi(t; 0, \mu_-), \qquad \nu_+(t) = \varphi(t; 0, \mu_+) \quad (t \in \mathbb{R}).$$

From

$$|\mu(t) - \nu_+(t)| = \left| \Phi_{t-T\lfloor \frac{t}{T} \rfloor}\left(\mu\left(T\left\lfloor \frac{t}{T} \right\rfloor\right)\right) - \Phi_{t-T\lfloor \frac{t}{T} \rfloor}(\mu_+) \right| \le \sup_{0\le s\le T} \left| \Phi_s\left(\mu\left(T\left\lfloor \frac{t}{T} \right\rfloor\right)\right) - \Phi_s(\mu_+) \right|$$

we see that $|\mu(t) - \nu_+(t)| \to 0$ as $t \to \infty$. Analogously, $|\mu(t) - \nu_-(t)| \to 0$ as $t \to -\infty$.

Assume now that μ is a forward attractor. Since, for any $\delta > 0$, $|\mu(t_0) - \nu_-(t_0)| < \delta$ for some sufficiently small t_0, we find

$$0 = \lim_{\tau\to\infty} |\varphi\,(t_0+\tau; t_0, \mu(t_0)) - \nu_-(t_0+\tau)| = \lim_{t\to\infty} |\mu(t) - \nu_-(t)|$$

and thus in particular $|\mu(kT) - \nu_-(kT)| = |\mu(kT) - \mu_-| \to 0$ as $k \to \infty$. Therefore $\mu_+ = \mu_-$, and μ is T-periodic.

Next assume that μ is a pullback attractor. Given $\delta > 0$, define $x_{0,\sigma}$ to be $\nu_-(\sigma)$ or $\mu(\sigma)$ depending on whether $|\mu(\sigma) - \nu_-(\sigma)| < \delta$ or not. Since, for all $t \in \mathbb{R}$,

$$0 = \lim_{\sigma\to-\infty} |\varphi(t; \sigma, x_{0,\sigma}) - \mu(t)| = |\nu_-(t) - \mu(t)|,$$

we find $\mu = \nu_-$, hence μ is T-periodic.

The remaining assertions are now obvious. $\square$

As in the autonomous case, attractors of a T-periodic equation can be characterised easily in terms of the associated Poincaré map Φ_T.

Theorem 23. *Assume (18) holds, and let $\mu : \mathbb{R} \to \mathbb{R}$ be a bounded solution of (3). Then the following statements are equivalent:*

(i) *μ is a forward attractor;*
(ii) *μ is a pullback attractor;*
(iii) *μ is a uniform attractor;*
(iv) *μ is T-periodic and $\mu(0)$ is an isolated fixed point of Φ_T such that $\Phi_T(a) - a > 0 > \Phi_T(b) - b$ for all $a < \mu(0) < b$ with $b - a$ sufficiently small.*

Proof. Let μ be a forward or pullback attractor. By Lemma 22, μ is T-periodic and hence $\mu(0)$ is a fixed point of Φ_T. If there existed a strictly increasing sequence (a_n) with $a_n \to \mu(0)$ and $\Phi_T(a_n) \le a_n$ for all n then $\Phi_T^k(a_n) \not\to \mu(0)$ as $k \to \infty$, so that μ could not be forward attracting. Setting $x_{0,\sigma} = a_n$ if σ/T is an integer, and $x_{0,\sigma} = \mu(\sigma)$ otherwise, we would have

$$\limsup_{k\to-\infty} |\varphi(0; kT, x_{0,kT}) - \mu(0)| \ge |a_n - \mu(0)| > 0$$

and thus μ would not be pullback attracting either. Overall, we see that (i) and (ii) both imply (iv).

Conversely, assume that $\Phi_T(a) - a > 0 > \Phi_T(b) - b$ for all $a < \mu(0) < b$ whenever $b - a < \varepsilon$. Pick $\delta > 0$ so small that $\bigcup_{0\le s\le T} \Phi_s^{-1}\,(]\mu(s)-\delta, \mu(s)+\delta[) \subset]\mu(0)-\varepsilon, \mu(0)+\varepsilon[$, i.e., whenever $|x - \mu(s)| < \delta$ for some s with $0 \le s \le T$ then also $|\Phi_s^{-1}(x) - \mu(0)| < \varepsilon$. Assume now that $\sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$. Observing

$$\begin{aligned} |\varphi(\sigma+\tau; \sigma, x_{0,\sigma}) - \mu(\sigma+\tau)| &= \left|\Phi_{s_2} \circ \Phi_T^k \circ \Phi_{s_1}^{-1}(x_{0,\sigma}) - \Phi_{s_2}\,(\mu(0))\right| \\ &\le \sup_{0\le s\le T} \left|\Phi_s \circ \Phi_T^k \circ \Phi_{s_1}^{-1}(x_{0,\sigma}) - \Phi_s\,(\mu(0))\right| \end{aligned}$$

with $s_2 = \sigma + \tau - T\lfloor(\sigma+\tau)/T\rfloor$, $s_1 = \sigma - T\lfloor\sigma/T\rfloor$ and $k = \lfloor(\sigma+\tau)/T\rfloor - \lfloor\sigma/T\rfloor$, and noting that $|\Phi_{s_1}^{-1}(x_{0,\sigma}) - \mu(0)| < \varepsilon$ for all $\sigma \in \mathbb{R}$, we deduce that $\|\varphi(\cdot+\tau; \cdot, x_{0,\cdot}) - \mu(\cdot+\tau)\|_\infty \to 0$ as $\tau \to \infty$. In other words, μ is a uniform attractor. $\square$

Corollary 24. *Assume (18) holds, and let $\mu : \mathbb{R} \to \mathbb{R}$ be a bounded solution of (3). Then μ is a uniform repellor if and only if μ is T-periodic and $\mu(0)$ is an isolated fixed point of Φ_T such that $\Phi_T(a) - a < 0 < \Phi_T(b) - b$ for all $a < \mu(0) < b$ with $b - a$ sufficiently small.*

Remark 25. According to Lemma 22 and Theorem 23 an attractor μ cannot be *sub*harmonic. It can, however, be *super*harmonic, i.e., the minimal period of μ can be smaller than T and thus of the form $\frac{1}{q}T$ for some $q \in \mathbb{N}$, $q \ge 2$. For a simple example consider

$$\dot{x} = 2\cos 2t + (1+\cos t)\sin 2t - (1+\cos t)x,$$

for which f has minimal period 2π, yet the global uniform attractor $\mu : t \mapsto \sin 2t$ is π-periodic.

For T-periodic functions f it is natural to also study the averaged version of (3), i.e., the autonomous equation

$$\dot{x} = \overline{f}(x) \quad \text{where } \overline{f}(x) = \frac{1}{T}\int_0^T f(t, x)\mathrm{d}t.$$

Note that $\overline{f}(x)$ is a polynomial whenever $f(t, x)$ is polynomial in x. Sometimes information about solutions of (3) can be obtained from the averaged equation, see for instance [39] for an introduction to the extensive theory of averaging. In the present context the averaging approach leads to

Theorem 26. *Assume $a_0, a_1, \ldots, a_d$ are continuous, T-periodic functions, and $\int_0^T a_d(t)\mathrm{d}t \neq 0$. Then, for every sufficiently small $\varepsilon > 0$, the total number of uniform attractors and repellors of*

$$\dot{x} = \varepsilon\left(a_0(t) + a_1(t)x + \cdots + a_d(t)x^d\right) \tag{19}$$

is at most d.

Proof. According to Theorem 21 the statement is true for $d \leq 2$, even without any assumptions on a_d and ε. We are therefore interested only in the case $d \geq 3$. For notational convenience we replace ε in (19) by ε^{d-1}. Thus modified, and rescaled via $y = \varepsilon x$, (19) takes the form

$$\dot{y} = a_d y^d + \varepsilon a_{d-1} y^{d-1} + \cdots + \varepsilon^{d-1} a_1 y + \varepsilon^d a_0. \tag{20}$$

We are going to show that (20) has at most d uniform attractors and repellors, provided that $\varepsilon > 0$ is sufficiently small. Without loss of generality assume that $\int_0^T a_d(t)\mathrm{d}t > 0$ as well as $a_d(0) > 0$.

For $\varepsilon = 0$ the solution of (20) with $y(0) = z$ is given by

$$y : t \mapsto \frac{z}{(1-(d-1)A_d(t)z^{d-1})^{\frac{1}{d-1}}}, \tag{21}$$

where $A_d(t) = \int_0^t a_d(s)\mathrm{d}s$, $0 \leq t \leq T$. This solution exists for all $0 \leq t \leq T$ and $z \in]-C_d^-, C_d^+[$ where, if d is odd,

$$0 < C_d^- = C_d^+ = \left((d-1)\max_{0\leq t\leq T} A_d(t)\right)^{-\frac{1}{(d-1)}} < \infty,$$

whereas for even d the value of C_d^+ remains unchanged but

$$0 < C_d^- = \left(-(d-1)\min_{0\leq t\leq T} A_d(t)\right)^{-\frac{1}{(d-1)}} \leq \infty.$$

(If $A_d(t) \geq 0$ for all t then C_d^- is understood to equal ∞.) Correspondingly, the Poincaré map associated with (20) can be written in the form

$$\Phi_T(z) = \frac{z}{(1-(d-1)A_d(T)z^{d-1})^{\frac{1}{d-1}}} + \varepsilon R(z, \varepsilon), \tag{22}$$

where, for every compact set $K \subset]-C_d^-, C_d^+[$ and ε sufficiently small, R depends continuously on (z, ε) and is in fact a (real-)analytic function of z. Moreover, uniformly on K,

$$R(z, \varepsilon) \to z^{d-1}\int_0^T a_{d-1}(t)\left(1-(d-1)A_d(t)z^{d-1}\right)^{\frac{1}{d-1}}\mathrm{d}t \quad \text{as } \varepsilon \to 0.$$

Similarly, integrating (20) in backward time, or inverting (22), yields

$$\Phi_T^{-1}(z) = \frac{z}{(1+(d-1)A_d(T)z^{d-1})^{\frac{1}{d-1}}} + \varepsilon S(z, \varepsilon),$$

a map analytic on $]-\widetilde{C}_d^-, \widetilde{C}_d^+[$ where, as before, for odd d,

$$0 < \widetilde{C}_d^- = \widetilde{C}_d^+ = \left((d-1)(\max_{0\leq t\leq T} A_d(t) - A_d(T))\right)^{-\frac{1}{(d-1)}} \leq \infty,$$

whereas for even d the number $\widetilde{C}_d^+$ remains unchanged, yet

$$0 < \widetilde{C}_d^- = \left((d-1)(A_d(T) - \min_{0\le t\le T} A_d(t))\right)^{-\frac{1}{(d-1)}} < \infty.$$

Note that, since $A_d(T) > 0$, we have $\widetilde{C}_d^+ > C_d^+$; also $\widetilde{C}_d^- > C_d^-$ or $\widetilde{C}_d^- < C_d^-$ depending on whether d is odd or even.

It is easy to see that for every $t_0 \in [0, T]$ with $a_d(t_0) \neq 0$ there exist positive numbers D_{t_0}, δ_{t_0} not depending on ε such that, for any bounded solution ν of (20),

$$|\nu(t)| \le D_{t_0} \quad \forall t : |t - t_0| < \delta_{t_0}.$$

Assume now that ν_ε is a T-periodic solution of (20). Since $a_d(0) > 0$ we clearly have $|\nu_\varepsilon(0)| \le D_0$ for all $\varepsilon > 0$. As a key step in our analysis, we are going to show that actually

$$\lim_{\varepsilon\to 0} \nu_\varepsilon(0) = 0. \tag{23}$$

To this end, pick $\nu_0 > C_d^+$ and note that, according to (21), the solution of (20) with $\varepsilon = 0$ and $y(0) = \nu_0$ becomes unbounded for some t^* with $0 < t^* \le T$. By increasing ν_0 slightly we may assume that $a_d(t^*) \neq 0$. Pick $\delta < \delta_{t^*}$ so small that $y(t^* - \delta) > D_{t^*}$. If, for some sequence (ε_n) with $\varepsilon_n \to 0$, we had $\nu_{\varepsilon_n}(0) \to \nu_0$, that is, if $\limsup_{\varepsilon\to 0} \nu_\varepsilon(0) \ge \nu_0$, then, by the continuous dependence on ε of the solutions of (20),

$$\nu_{\varepsilon_n}(t^* - \delta) \to y(t^* - \delta) > D_{t^*} \quad \text{as } n \to \infty.$$

Clearly this would contradict the fact that, for all n,

$$|\nu_{\varepsilon_n}(t)| \le D_{t^*} \quad \forall t : |t - t^*| < \delta_{t^*}.$$

Hence $\limsup_{\varepsilon\to 0} \nu_\varepsilon(0) < \nu_0$, and, since $\nu_0 > C_d^+$ was arbitrary, we deduce $\limsup_{\varepsilon\to 0} \nu_\varepsilon(0) \le C_d^+$. Completely similar reasoning, using the fact that (21) becomes unbounded in forward (if d is odd) or backward time (if d is even) whenever $y(0) < -\widetilde{C}_d^-$, shows that $\liminf_{\varepsilon\to 0} \nu_\varepsilon(0) \ge -\widetilde{C}_d^-$. To see that the inequalities in these relations are strict assume that $\nu_{\varepsilon_n}(0) \to C_d^+$. Then

$$\nu_{\varepsilon_n}(0) = \Phi_T^{-1}\left(\nu_{\varepsilon_n}(0)\right) \to \frac{C_d^+}{(1 + (d-1)A_d(T)(C_d^+)^{d-1})^{\frac{1}{d-1}}} < C_d^+,$$

an obvious contradiction; for $\nu_{\varepsilon_n}(0) \to -\widetilde{C}_d^-$ considering $\nu_{\varepsilon_n}(0) = \Phi_T^{-1}\left(\nu_{\varepsilon_n}(0)\right)$ if d is odd, and $\nu_{\varepsilon_n}(0) = \Phi_T\left(\nu_{\varepsilon_n}(0)\right)$ if d is even, leads to a similar contradiction. We deduce from all this that

$$-\widetilde{C}_d^- < \liminf_{\varepsilon\to 0} \nu_\varepsilon(0) \le \limsup_{\varepsilon\to 0} \nu_\varepsilon(0) < C_d^+.$$

Now the above argument can be iterated. Let $\nu_{\varepsilon_n}(0) \to a$ and assume first that $0 \le a < C_d^+$. Then

$$a = \lim_{n\to\infty} \nu_{\varepsilon_n}(0) = \lim_{n\to\infty} \Phi_T^{-1}\left(\nu_{\varepsilon_n}(0)\right) = \frac{a}{(1 + (d-1)A_d(T)a^{d-1})^{\frac{1}{d-1}}},$$

and hence $a = 0$; similarly, if $-\widetilde{C}_d^- < a \le 0$ then $a = 0$. Thus we have established (23).

To conclude the proof, let $\nu_\varepsilon^{(1)}, \ldots, \nu_\varepsilon^{(L)}$ be L different uniform attractors or repellors of (20) and pick $C < \min(\widetilde{C}_d^-, C_d^+)$. Then $|\nu_\varepsilon^{(i)}(0)| < C$ for all $i = 1, \ldots, L$ and all sufficiently small ε. When considered a function of the *complex* variable z, Φ_T according to (22) is, for small ε, analytic on (an open set containing the closure of) the open disc $B_C = \{z : |z| < C\}$ and, uniformly on B_C,

$$\Phi_T(z) - z \to \frac{z}{(1 - (d-1)A_d(T)z^{d-1})^{\frac{1}{d-1}}} - z =: G(z) \quad \text{as } \varepsilon \to 0.$$

The function G is analytic on $B_{\min(\widetilde{C}_d^-, C_d^+)}$ and does not vanish on the boundary of B_C. By Hurwitz's Theorem (see e.g. [37]) there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the functions $\Phi_T - \mathrm{id}_{\mathbb{C}}$ and G have the same number of zeros in B_C, where all zeros are counted according to their multiplicity. Clearly, $G(z) = 0$ implies $z = 0$, and from

$$G(z) = \frac{z}{(1-(d-1)A_d(T)z^{d-1})^{\frac{1}{d-1}}} - z = A_d(T)z^d + \mathcal{O}(z^{2d-1}),$$

we see that $z = 0$ is a d-fold zero of G. Consequently, for $\varepsilon < \varepsilon_0$, the Poincaré map Φ_T has exactly d *complex* fixed points; as a function of the *real* argument z, Φ_T has *at most* d fixed points. Hence $L \le d$ and the proof is complete. □

Remark 27. (i) The assumption $\int_0^T a_d(t)\mathrm{d}t \neq 0$ in Theorem 26 ensures that the averaged right-hand side in (19) has actual degree d. In this case, and if μ_ε is a uniform attractor or repellor of (19) such that $\varepsilon \mapsto \mu_\varepsilon(0)$ is continuous, one can show that, uniformly in t,

$$\mu_\varepsilon(t) \to \mu_0 \quad \text{as } \varepsilon \to 0,$$

where μ_0 is an equilibrium of the averaged equation, that is, $\overline{p}(\mu_0) = 0$ with

$$\overline{p}(x) = \int_0^T a_0(t)\mathrm{d}t + x\int_0^T a_1(t)\mathrm{d}t + \cdots + x^d\int_0^T a_d(t)\mathrm{d}t.$$

An outline of the argument which extends [40, Thm. 3.1] is as follows: A lengthy yet elementary calculation yields a refined version of (22) for the representation of the Poincaré map Φ_T associated with (20), namely

$$\Phi_T(z) = \frac{z}{(1-(d-1)A_d(T)z^{d-1})^{\frac{1}{d-1}}} + \sum_{i=1}^{d} \varepsilon^i R_i(z) + \varepsilon^{d+1}\widetilde{R}(z,\varepsilon), \tag{24}$$

where

$$R_i(z) = z^{d-i}\int_0^T a_{d-i}(t)\mathrm{d}t + \mathcal{O}(z^{2d-2i+1}) \quad (i = 1,\ldots,d),$$

and, as before, the remainder term $\widetilde{R}(z,\varepsilon)$ is analytic in z and uniformly convergent as $\varepsilon \to 0$. In terms of the original coordinate x, representation (24) has the form

$$\Phi_T(\varepsilon x) = \varepsilon x + \varepsilon^d \overline{p}(x) + \varepsilon^{d+1}\widetilde{R}(\varepsilon x, \varepsilon).$$

If μ_ε is a periodic solution of (19), again with ε replaced by ε^{d-1}, then

$$\varepsilon\mu_\varepsilon(0) = \Phi_T(\varepsilon\mu_\varepsilon(0)) = \varepsilon\mu_\varepsilon(0) + \varepsilon^d\overline{p}(\mu_\varepsilon(0)) + \varepsilon^{d+1}\widetilde{R}(\varepsilon\mu_\varepsilon(0), \varepsilon)$$

and hence, for all $\varepsilon > 0$,

$$\overline{p}(\mu_\varepsilon(0)) + \varepsilon\widetilde{R}(\varepsilon\mu_\varepsilon(0), \varepsilon) = 0.$$

Since the proof of Theorem 26 has shown that $\varepsilon\mu_\varepsilon(0) \to 0$ as $\varepsilon \to 0$, it follows that $\overline{p}(\mu_\varepsilon(0)) \to 0$ as well. Therefore, if $\varepsilon \mapsto \mu_\varepsilon$ is a continuous parametrisation then $\mu_\varepsilon \to \mu_0$ with $\overline{p}(\mu_0) = 0$.

(ii) An explicit (upper) bound for ε can be derived from an application of Rouché's Theorem, see [35] for a somewhat similar result.

(iii) It is intriguing to speculate whether Theorems 23 and 26 might have any analogues for *almost periodic* equations. In general an attractor of (3) can be *much* more complicated if $f = f(t,x)$ is merely almost periodic in t. However, it has been demonstrated in [41] that a form of uniformity (which, though different, does bear some similarity with the uniformity discussed here) greatly reduces the possible complexity of attractors.

We shall finally demonstrate that neither of the two assumptions in Theorem 26, namely smallness of ε and preservation of the degree d under averaging, can be discarded easily. To see the importance of the smallness of ε, recall from Theorem 21 that the size of ε is irrelevant for the assertion in Theorem 26 as long as $d \le 2$, and also if $d = 3$ and a_3 does not change its sign. If, however, a_3 does change its sign, then Theorem 26 generally fails for larger ε, even if $\int_0^T a_3(t)\mathrm{d}t$ happens to be nonzero. This is the content of the following theorem which exploits an ingenious idea in [30].

Theorem 28. *Let $a_3 : \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic, and assume a_3 does change its sign, i.e., $a_3(s)a_3(t) < 0$ for some $s, t \in \mathbb{R}$. Then, given $N \in \mathbb{N}$, there exists a smooth, T-periodic function a_2 such that*

$$\dot{x} = a_2 x^2 + a_3 x^3 \tag{25}$$

has N uniform attractors.

Proof. We are going to construct a smooth, T-periodic function a such that

$$\dot{y} = a y^2 + \varepsilon^2 a_3 y^3 \tag{26}$$

has N uniform attractors, provided that $\varepsilon > 0$ is sufficiently small. This will prove the theorem because setting $a_2 = a/\varepsilon$ and rescaling (26) via $x = \varepsilon y$ yields (25).

Given a_3 with the stated properties it remains to find a such that, for small ε, (26) has N uniform attractors. To this end, let $A(t) = \int_0^t a(s)\mathrm{d}s$ and observe that the solution of (26) for $\varepsilon = 0$ and $y(0) = z$ is given by

$$\varphi(t; 0, z) = \frac{z}{1 - zA(t)}.$$

We will choose $A \neq 0$ such that $A(0) = A(T) = 0$. Thus every solution of (26) for $\varepsilon = 0$ is T-periodic provided that $|z| < (\max_{0\le t\le T} |A(t)|)^{-1} =: \gamma$. The Poincaré map associated with (26) can be written in the form

$$\Phi_T(z) = z + \varepsilon^2 R(z) + \varepsilon^3 S(z, \varepsilon), \tag{27}$$

where $R(z) = z^3 Q_A(z)$ and

$$Q_A(z) = \int_0^T \frac{a_3(t)}{1 - zA(t)}\mathrm{d}t,$$

and S depends smoothly on z and ε. Assume that $z_0 \neq 0$ is a simple zero of R, i.e., $R(z_0) = 0$ and $R'(z_0) \neq 0$. Then, for every $\varepsilon > 0$ sufficiently small, there exists z_ε such that $R(z_\varepsilon) + \varepsilon S(z_\varepsilon, \varepsilon) = 0$ and hence $\Phi_T(z_\varepsilon) = z_\varepsilon$. Moreover,

$$\Phi_T'(z_\varepsilon) = 1 + \varepsilon^2 R'(z_\varepsilon) + \varepsilon^3 \frac{\partial}{\partial z} S(z_\varepsilon, \varepsilon) \neq 1,$$

so that $t \mapsto \varphi(t; 0, z_\varepsilon)$ is, by Theorem 23, a uniform attractor or repellor of (26). Note that z_0 is a zero of Q_A also, and $R'(z_0) = z_0^3 Q_A'(z_0)$. The proof will therefore be complete once we can choose A in such a way that Q_A has N distinct zeros $\zeta_1, \ldots, \zeta_N$ with $Q_A'(\zeta_i) < 0$ and $0 < \zeta_i < \gamma$ for all $i = 1, \ldots, N$. To this end, and denoting the indicator function of an arbitrary set $M \subset \mathbb{R}$ by $\mathbf{1}_M$, we first use the (non-smooth) ansatz

$$A(t) = \sum_{i=1}^{2N} \alpha_i \mathbf{1}_{I_i}(t) \quad (0 \le t \le T), \tag{28}$$

where the positive numbers α_i and the compact intervals $I_i \subset]0, T[$ will be chosen inductively below. In a final step we will afterwards approximate A from below by a C^∞ function $\widetilde{A}$, and $a = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{A}$ will have all the properties required.

Assume for the time being that $\int_0^T a_3(t)\mathrm{d}t > 0$. Pick $2N$ *different* points $0 < t_1, \ldots, t_{2N} < T$ such that

$$(-1)^i a_3(t_i) > 0 \quad \forall i = 1, \ldots, 2N. \tag{29}$$

Define $t_0 = 0$ and $t_{2N+1} = T$ and let $\delta = \frac{1}{2}\min\{|t_j - t_k| : j, k = 0, \ldots, 2N+1;\ j \neq k\}$; also define functions $Q^{(1)}, \ldots, Q^{(2N)}$ as

$$Q^{(i)}(z) = \int_0^T a_3(t)\mathrm{d}t + \sum_{j=1}^{i} \frac{\alpha_j z}{1 - \alpha_j z} \int_{I_j} a_3(t)\mathrm{d}t \quad (i = 1, \ldots, 2N),$$

so that in particular $Q^{(2N)} = Q_A$ with A according to (28). Set $\alpha_1 = 1$, and let $I_1 = [t_1 - \frac{1}{2}\delta_1, t_1 + \frac{1}{2}\delta_1]$ with $\delta_1 < \delta$ so small that $\int_{I_1} a_3(t)\mathrm{d}t < 0$. There exists a unique number $\zeta_1^{(1)}$ with $0 < \zeta_1^{(1)} < 1$ such that $Q^{(1)}(\zeta_1^{(1)}) = 0$ yet $\frac{\mathrm{d}}{\mathrm{d}z} Q^{(1)}(\zeta_1^{(1)}) \neq 0$ (see Fig. 4), i.e., $\zeta_1^{(1)}$ is a simple root of $Q^{(1)}$. Next let $\alpha_2 = 3/(\zeta_1^{(1)} + 2)$

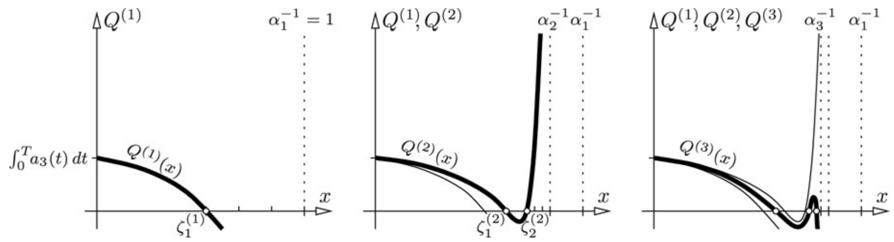

Fig. 4. Visualising the first steps in the proof of Theorem 28.

and $I_2 = [t_2 - \frac{1}{2}\delta_2, t_2 + \frac{1}{2}\delta_2]$ where $\delta_2 < \delta$ is chosen sufficiently small to ensure that $\int_{I_2} a_3(t)\mathrm{d}t > 0$ as well as $Q^{(2)}(\frac{2}{3}\zeta_1^{(1)} + \frac{1}{3}) < 0$. Consequently, there exist two numbers $\zeta_1^{(2)}, \zeta_2^{(2)}$ with $0 < \zeta_1^{(2)} < \frac{2}{3}\zeta_1^{(1)} + \frac{1}{3} < \zeta_2^{(2)} < \alpha_2^{-1}$ such that $Q^{(2)}(\zeta_1^{(2)}) = Q^{(2)}(\zeta_2^{(2)}) = 0$. Obviously, $Q^{(2)}$ has no other roots apart from $\zeta_1^{(2)}, \zeta_2^{(2)}$, and the latter are simple. Assume that $1 = \alpha_1 < \alpha_2 < \cdots < \alpha_i$ as well as $\delta_1, \ldots, \delta_i < \delta$ have been chosen inductively such that $Q^{(i)}$ has i simple roots $0 < \zeta_1^{(i)} < \cdots < \zeta_i^{(i)} < \alpha_i^{-1}$. Let $\alpha_{i+1} = 3/(\zeta_i^{(i)} + 2\alpha_i^{-1})$ and $I_{i+1} = [t_{i+1} - \frac{1}{2}\delta_{i+1}, t_{i+1} + \frac{1}{2}\delta_{i+1}]$. The number $\delta_{i+1} < \delta$ is chosen so small that $(-1)^{i+1}\int_{I_{i+1}} a_3(t)\mathrm{d}t > 0$ and $(-1)^i Q^{(i+1)}(\frac{2}{3}\zeta_i^{(i)} + \frac{1}{3}\alpha_{i+1}^{-1}) > 0$ but also

$$(-1)^j Q^{(i+1)}\left(\frac{1}{2}\zeta_j^{(i)} + \frac{1}{2}\zeta_{j+1}^{(i)}\right) > 0 \quad \forall j = 1, \ldots, i-1.$$

The function $Q^{(i+1)}$ has $i+1$ zeros $\zeta_1^{(i+1)}, \ldots, \zeta_{i+1}^{(i+1)}$ with

$$0 < \zeta_1^{(i+1)} < \frac{1}{2}\zeta_1^{(i)} + \frac{1}{2}\zeta_2^{(i)} < \zeta_2^{(i+1)} < \cdots < \frac{2}{3}\zeta_i^{(i)} + \frac{1}{3}\alpha_{i+1}^{-1} < \zeta_{i+1}^{(i+1)} < \alpha_{i+1}^{-1}.$$

Since $Q^{(i+1)}$ is the sum of $i+1$ rational functions of degree one and therefore can have *at most* $i+1$ zeros, all the numbers $\zeta_j^{(i+1)}$ are simple zeros, in fact,

$$(-1)^j \frac{\mathrm{d}}{\mathrm{d}z} Q^{(i+1)}(\zeta_j^{(i+1)}) > 0 \quad \forall j = 1, \ldots, i+1.$$

By carrying out $2N$ steps of the above procedure (see also Fig. 4) we obtain a function $Q = Q^{(2N)}$ having N different zeros $\zeta_1^{(2N)}, \zeta_3^{(2N)}, \ldots, \zeta_{2N-1}^{(2N)}$ with $Q'(\zeta_{2i-1}^{(2N)}) < 0$ for all $i = 1, \ldots, N$. Clearly

$$\zeta_i^{(2N)} \le \zeta_{2N}^{(2N)} < \alpha_{2N}^{-1} = \gamma \quad \forall i = 1, \ldots, 2N,$$

hence the usage of (27) is justified. Finally, to produce a smooth function a, let A_n be an increasing sequence of smooth, non-negative functions such that for all $0 \le t \le T$, $0 \le A_n(t) \le A(t)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} A_n(t) = A(t)$. Pick C between $\zeta_{2N}^{(2N)}$ and γ. By dominated convergence, $\lim_{n\to\infty} Q_{A_n}(z) = Q_A(z)$ uniformly for $0 \le z \le C$. For sufficiently large n therefore Q_{A_n} has N different simple zeros with negative slope as well. Extending A_n to a smooth T-periodic function $\widetilde{A} : \mathbb{R} \to \mathbb{R}$ and setting $a = \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{A}$ completes the proof for the case $\int_0^T a_3(t)\mathrm{d}t > 0$.

If $\int_0^T a_3(t)\mathrm{d}t < 0$ then completely analogous reasoning applies with all signs inverted, e.g., (29) has to be replaced by

$$(-1)^i a_3(t_i) < 0 \quad \forall i = 1, \ldots, 2N \text{ etc.}$$

Finally, if $\int_0^T a_3(t)\mathrm{d}t = 0$ then $Q^{(1)}$ will not have a zero except for $z = 0$. Therefore, $2N+1$ rather than $2N$ points t_i have to be chosen; apart from this modification the same procedure as above applies. Overall, the proof is complete. □

As a demonstration that Theorem 21 cannot be extended to any d larger than three, even with a definiteness assumption for a_d, we provide

Corollary 29. *Let $d \geq 4$ be a natural number, and assume $a_d : \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic and either non-negative or non-positive. Then, given $N \in \mathbb{N}$, there exist smooth, T-periodic functions a_2, a_3 such that*

$$\dot{x} = a_2 x^2 + a_3 x^3 + a_d x^d \tag{30}$$

has N uniform attractors.

Proof. Up to and including terms of order ε^2, the Poincaré map Φ_T associated with

$$\dot{y} = a y^2 + \varepsilon^2 a_3 y^3 + \varepsilon^{d-1} a_d y^d \tag{31}$$

is identical with the one in the previous proof, that is, (27) holds with only the remainder term S modified. Thus, taking as a_3 any smooth, T-periodic function which changes its sign and subsequently constructing a as in the proof of Theorem 28, we conclude that (31) has N uniform attractors whenever $\varepsilon > 0$ is sufficiently small. Setting $a_2 = a/\varepsilon$ and rescaling (31) via $x = \varepsilon y$ yields (30). □

Our final result highlights the role of the non-degeneracy condition $\int_0^T a_d(t)\mathrm{d}t \neq 0$ in Theorem 26 by showing that the latter generally fails if this condition is not met.

Corollary 30. *Let $d \geq 3$ be a natural number and $a_d \neq 0$ a continuous, T-periodic function with $\int_0^T a_d(t)\mathrm{d}t = 0$. Then, given $N \in \mathbb{N}$, there exists a smooth T-periodic function a_2 such that, for all sufficiently small $\varepsilon > 0$,*

$$\dot{x} = \varepsilon(a_2 x^2 + a_{\mathrm{d}} x^d) \tag{32}$$

has N uniform attractors.

Proof. The argument is quite similar to the one proving Theorem 28. Clearly, the assertion remains unchanged if, for notational convenience, ε is replaced by ε^{d-1} in (32). Thus modified, and rescaled via $x = y/\varepsilon$, (32) takes the form

$$\dot{y} = \varepsilon^{d-2} a_2 y^2 + a_d y^d. \tag{33}$$

The proof will therefore be complete once we demonstrate how to find a_2 not depending on ε such that (33) has N uniform attractors or repellors for all sufficiently small $\varepsilon > 0$.

Let $A_d = \int_0^t a_d(s)\mathrm{d}s$ and note that $A_d(0) = A_d(T) = 0$. Similarly to (27) the Poincaré map Φ_T associated with (33) can be written as

$$\Phi_T(z) = z + \varepsilon^{d-2} R(z) + \varepsilon^{d-1} S(z, \varepsilon) \tag{34}$$

with $R(z) = z^2 Q_{a_2}(z)$ and

$$Q_{a_2}(z) = \int_0^T a_2(t)\left(1 - (d-1)A_d(t) z^{d-1}\right)^{\frac{d-2}{d-1}} \mathrm{d}t,$$

provided that $|z| < C_d$, where

$$C_d = \left((d-1)\max_{0 \leq t \leq T} |A_d(t)|\right)^{-\frac{1}{d-1}} > 0.$$

As before, it only remains to choose a_2 in such a way that Q_{a_2} has N simple zeros within the interval $]0, C_d[$.

Obviously, Q_{a_2} depends linearly upon a_2, and, for $|z| < C_d$,

$$Q_{a_2}(z) = \sum_{k=0}^{\infty} \binom{\frac{d-2}{d-1}}{k} (1-d)^k z^{k(d-1)} \int_0^T a_2(t) A_d^k(t)\mathrm{d}t.$$

Thus, if a_2 is orthogonal to $\mathbf{1}$ $(=A_d^0)$, $A_d, \ldots, A_d^{i-1}$ yet not orthogonal to A_d^i then the leading term of $Q_{a_2}(z)$ will be of the order $z^{i(d-1)}$. More formally, choose N smooth, T-periodic functions $a_2^{(i)}$, $i = 1, \ldots, N$ such that, for all i,

$$\int_0^T a_2^{(i)}(t)\mathrm{d}t = \int_0^T a_2^{(i)}(t) A_d(t)\mathrm{d}t = \cdots = \int_0^T a_2^{(i)}(t) A_d^{i-1}(t)\mathrm{d}t = 0 \quad \text{yet} \quad \int_0^T a_2^{(i)}(t) A_d^i(t)\mathrm{d}t \neq 0.$$

(Such a choice is possible because A_d is continuous but not constant.). With this,

$$Q_{a_2^{(i)}}(z) = q_i z^{i(d-1)} + \mathcal{O}(z^{(i+1)(d-1)}) \quad (i = 1, \dots, N),$$

with constants $q_i \neq 0$. From (a slightly generalised version of) [30, Lem. 3.1] it follows that there exist positive numbers $\gamma_1, \dots, \gamma_N$ such that with

$$a_2 = \gamma_N a_2^{(N)} - \gamma_{N-1} a_2^{(N-1)} + \cdots + (-1)^{N-1} \gamma_1 a_2^{(1)}$$

the function Q_{a_2} has N simple zeros in $]0, C_d[$. □

4. Concluding remarks

The success of classical bifurcation theory for autonomous differential equations $\dot{x} = F(x; \lambda)$ which depend on a parameter λ is due to several facts which, albeit obvious, have to be reconsidered carefully when a nonautonomous bifurcation theory is to be developed:

(i) Many relevant models in applied sciences are indeed of the form $\dot{x} = F(x; \lambda)$;
(ii) Potential bifurcating objects like equilibria and periodic orbits are *characterised algebraically* as zeros of F and a Poincaré map, respectively;
(iii) Usually, there are not too many bifurcating objects, and the generic ones can to some extent be classified [2];
(iv) Asymptotic stability of equilibria and periodic orbits can be determined through *linear algebra* by means of, respectively, the linearisation of F and a Poincaré map;
(v) The *transient* behaviour of solutions is related to their *asymptotic* behaviour; more precisely, the long-term dynamics of an attractive equilibrium or periodic orbit can be approximated arbitrarily well by the transient behaviour on longer and longer time intervals. In particular, finite time numerics can be used to approximate asymptotic behaviour.

The results in this article show that in many respects the nonautonomous situation remains fundamentally different from the autonomous one, and the simple facts provided here allow for a clear understanding as to why this is so. Concretely, when reconsidering the above list in an nonautonomous setting one is lead naturally to a variety of fundamental questions which we feel are largely open at present.

$(\text{i})_t$ What are the characteristics of a *prototypical nonautonomous model* $\dot{x} = f(t, x; \lambda)$? How can one choose the appropriate time dependence for f? So far, we have studied nonautonomous attraction and repulsion only in its simplest uniform version for scalar equations. Already an astonishing dynamical complexity emerges which is in a sense due to the lack of better knowledge about how precisely f depends on t or, more positively put, the amazing "degree of freedom" which comes with time-varying vector fields. For example, our analysis of periodic equations does not incorporate the rate $\frac{\partial}{\partial t} f$ at which f is changing in time. On the other hand, this rate is known to play a prominent role for instance in explicit conditions ensuring asymptotic stability (albeit in a slightly different context, see [11]). We feel it necessary to push for the development, driven by applications, of a methodology of *nonautonomous modelling*.

$(\text{ii})_t$ Uniformly attracting or repelling solutions *cannot be detected algebraically*, e.g. by looking for zeros of f. Rather, the asymptotic behaviour and the spatial structure of $(1)_t$ tend to be interwoven in an intricate manner. Theorems 17 and 26 may serve as simple but perhaps prototypical examples. In fact, we expect that most questions regarding uniform attractors will touch upon a delicate balance between temporal and spatial complexity (see also [28]). Versatile, effective tools for analysing this balance have yet to be developed.

$(\text{iii})_t$ In view of Theorems 26 and 28 it appears doubtful whether the clear-cut situation for autonomous equations (i.e., few, classifiable bifurcation scenarios) does have an analogue in the nonautonomous setting. If so, a satisfactory treatment of the classification problem is desirable; if not, a bewildering zoo of bifurcation phenomena is likely to await exploration.

$(\text{iv})_t$ Attraction or repulsion of a solution can in general *not be determined directly from the coefficients* of the linearisation (see e.g. the instructive example in [28]). To comprehensively cover uniform attractors and repellors the existing asymptotic methods for nonautonomous equations will need substantial refinement.

$(\mathrm{v})_t$ What is the transient behaviour of a uniformly attracting or repelling solution, in particular, *how repelling can a uniformly attracting solution be* on an arbitrary bounded interval in time? We argue that this question is relevant not only for obvious practical reasons (as for instance no physical observation can cover an infinite interval in time) but also for mathematical reasons. In view of the enormous freedom introduced via an explicit yet unspecified time dependence in $(1)_t$ it is plausible that quite generally asymptotic concepts are of limited use and finite time concepts may step in.

In the context of $(\mathrm{v})_t$ we mention in closing a recent approach [5,12] that avoids any assumptions on the time dependence of f and that instead studies the transient behaviour of $(1)_t$ on intervals of finite time by exploiting all available (numerical) information. A solution x of $(1)_t$ on a finite time interval $[a, b]$ is *attracting* if $S(t) := \frac{\partial}{\partial x} f(t, x(t))$, or more generally the symmetric matrix $S(t) := \frac{1}{2}\left[\frac{\partial}{\partial x} f(t, x(t)) + \frac{\partial}{\partial x} f(t, x(t))^T\right]$ if $x \in \mathbb{R}^n$, $n \geq 2$, is negative (definite) for all $t \in [a, b]$. As a consequence, if x is attracting then all solutions which are close to x have a decreasing distance from x on $[a, b]$; analogously, x is *repelling* if $S(t)$ is positive (definite) for all $t \in [a, b]$. It is easy to see that the polynomial equation (16) with arbitrary time dependence on $[a, b]$ has at most d connected components in $[a, b] \times \mathbb{R}$ such that each component consists entirely of attracting or repelling solutions. Observations like these suggest that finite time concepts may indeed complement asymptotic concepts and may thus contribute towards a better understanding of nonautonomous differential equations.

Acknowledgements

The first author was partially supported by a HUMBOLDT Research Fellowship; he wishes to express his gratitude to the Johann Wolfgang Goethe-Universität Frankfurt for the hospitality he enjoyed during his visits in 2005 and 2006. The second author was supported by the EMMY NOETHER Program funded by the Deutsche Forschungsgemeinschaft.

References

[1] M.A.M. Alwash, Periodic solutions of polynomial non-autonomous differential equations, Electron. J. Differential Equations 84 (2005) 1–8.
[2] V.I. Arnold, Dynamical Systems V. Bifurcation Theory and Catastrophe Theory, in: Encyclopedia of Mathematical Sciences, vol. V, Springer, Berlin, Heidelberg, New York, 1994.
[3] B. Aulbach, T. Wanner, Integral manifolds for Carathéodory type differential equations in Banach spaces, in: B. Aulbach, F. Colonius (Eds.), Six Lectures on Dynamical Systems, World Scientific, Singapore, 1996.
[4] B. Aulbach, T. Wanner, The Hartman–Grobman theorem for Carathéodory-type differential equations in Banach spaces, Nonlinear Anal. TMA 40 (2000) 91–104.
[5] A. Berger, T.S. Doan, S. Siegmund, Nonautonomous finite-time dynamics, 2007. Preprint.
[6] D.N. Cheban, P.E. Kloeden, B. Schmalfuss, The relationship between pullback, forward and global attractors of nonautonomous dynamical systems, Nonlinear Dyn. Syst. 2 (2002) 125–144.
[7] H. Crauel, A. Debusche, F. Flandoli, Random attractors, J. Dynam. Differential Equations 9 (1997) 307–341.
[8] H. Crauel, F. Flandoli, Attractors for random dynamical systems, Probab. Theory Related Fields 100 (1994) 365–393.
[9] H. Crauel, P. Imkeller, M. Steinkamp, Bifurcations of one-dimensional differential equations, in: H. Crauel, V.M. Gundlach (Eds.), Stochastic Dynamics, Springer, 1999, pp. 27–47.
[10] J. Devlin, Periodic solutions of polynomial non-autonomous differential equations, Proc. Soc. Roy. Edinburgh Sect. A 123 (1993) 783–801.
[11] L.H. Duc, A. Ilchmann, S. Siegmund, P. Taraba, On stability of linear time-varying second-order differential equations, Quart. Appl. Math. 64 (2006) 137–151.
[12] L.H. Duc, S. Siegmund, Hyperbolicity and invariant manifolds for planar nonautonomous systems on finite time intervals, 2006. Preprint.
[13] R. Fabbri, R.A. Johnson, F. Mantellini, A nonautonomous saddle-node bifurcation pattern, Stoch. Dyn. 4 (2004) 335–350.
[14] J. Hale, H. Koçak, Dynamics and Bifurcations, Springer, New York, 1991.
[15] M.C. Irwin, Smooth Dynamical Systems, Academic Press, New York, London, 1980.
[16] R.A. Johnson, An application of topological dynamics to bifurcation theory, in: M.G. Nerurkar, D.P. Dokken, D.B. Ellis (Eds.), Topological Dynamics and Applications. A Volume in Honor of Robert Ellis, in: Contemporary Mathematics, vol. 215, American Mathematical Society, Providence, Rhode Island, 1998, pp. 323–334.
[17] R.A. Johnson, P.E. Kloeden, R. Pavani, Two-step transition in nonautonomous bifurcations: An explanation, Stoch. Dyn. 2 (2002) 67–92.
[18] R.A. Johnson, F. Mantellini, A nonautonomous transcritical bifurcation problem with an application to quasi-periodic bubbles, Discrete Contin. Dyn. Syst. 9 (2003) 209–224.
[19] R.A Johnson, Y. Yi, Hopf bifurcation from non-periodic solutions of differential equations, II, J. Differential Equations 107 (1994) 310–340.
[20] O.G. Khudaĭberenov, N.O. Khudaĭverenov, An estimate for the number of periodic solutions of the Abel equation, Differ. Uravn. 8 (2004) 1140–1142.
[21] P.E. Kloeden, Pullback attractors in nonautonomous difference equations, J. Differ. Equations Appl. 6 (2000) 33–52.
[22] P.E. Kloeden, Pitchfork and transcritical bifurcations in systems with homogeneous nonlinearities and an almost periodic time coefficient, Commun. Pure Appl. Anal. 3 (2004) 161–173.

[23] P.E. Kloeden, H. Keller, B. Schmalfuß, Towards a theory of random numerical dynamics, in: H. Crauel, V.M. Gundlach (Eds.), Stochastic Dynamics, Springer, 1999, pp. 259–282.

[24] P.E. Kloeden, B. Schmalfuß, Nonautonomous systems, cocycle attractors and variable time-step discretization, Numer. Algorithms 14 (1997) 141–152.

[25] P.E. Kloeden, S. Siegmund, Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 5 (2005) 1–21.

[26] J.A. Langa, J.C. Robinson, A. Rodriguez-Bernal, A. Suarez, A. Vidal-Lopez, Existence and nonexistence of unbounded forwards attractor for a class of nonautonomous reaction diffusion equations, Discrete Contin. Dyn. Syst. 18 (2007) 483–497.

[27] J.A. Langa, J.C. Robinson, A. Suarez, Stability, instability, and bifurcation phenomena in non-autonomous differential equations, Nonlinearity 15 (2002) 887–903.

[28] J.A. Langa, J.C. Robinson, A. Suarez, Bifurcations in non-autonomous scalar equations, J. Differential Equations 221 (2006) 1–35.

[29] D. Li, P.E. Kloeden, Equi-attraction and the continuous dependence of pullback attractors on parameters, Stoch. Dyn. 4 (2004) 373–384.

[30] A. Lins Neto, On the number of solutions of the equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{j=0}^{n} a_j(t)x^j$, $0 \le t \le 1$, for which $x(0) = x(1)$, Invent. Math. 59 (1980) 67–76.

[31] N.G. Lloyd, The number of periodic solutions of the equation $\dot{z} = z^N + p_1(t)z^{N-1} + \cdots + p_N(t)$, Proc. London Math. Soc. 27 (1973) 667–700.

[32] I. Malta, N.C. Saldanha, C. Tomei, Morin singularities and global geometry in a class of ordinary differential operators, Topol. Methods Nonlinear Anal. 10 (1997) 137–169.

[33] L. Markus, Asymptotically autonomous differential systems, Annal. Math. Studies 3 (1956) 17–29.

[34] K.J. Palmer, A generalization of Hartman's linearization theorem, J. Math. Anal. Appl. 41 (1973) 753–758.

[35] A.A. Panov, The number of periodic solutions of polynomial differential equations, Math. Notes 64 (1998) 622–628.

[36] M. Rasmussen, Attractivity and Bifurcations for Nonautonomous Dynamical Systems, in: Lecture Notes in Mathematics, vol. 1907, Springer, New York, 2007.

[37] R. Remmert, Theory of Complex Functions, Springer, New York, 1991.

[38] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differential Equations 27 (1978) 320–358.

[39] J.A. Sanders, F. Verhulst, Averaging Methods in Nonlinear Dynamical Systems, Springer, New York, 1985.

[40] S. Shashahani, Periodic solutions of polynomial first order differential equations, Nonlinear Anal. TMA 5 (1981) 157–165.

[41] W. Shen, Y. Yi, Almost automorphic and almost periodic dynamics in skew-product semiflows, Mem. Amer. Math. Soc. 136 (647) (1998).

[42] S. Siegmund, Dichotomy spectrum for nonautonomous differential Equations, J. Dynam. Differential Equations 14 (2002) 243–258.

[43] A.N. Sositaisvili, Bifurcations of topological type at singular points of parametrized vector fields, Funct. Anal. Appl. 6 (1972) 169–170. Translation from Funkts. Anal. Prilozh. 6 97–98.

[44] H.R. Thieme, Convergence results and a Poincaré–Bendixson trichotomy for asymptotically autonomous differential equations, J. Math. Biol. 30 (1992) 755–763.

[45] H.R. Thieme, Asymptotically autonomous differential equations in the plane, Rocky Mountain J. Math. 24 (1994) 351–380.

[46] P.J. Torres, Existence of closed solutions for a polynomial first order differential equation, J. Math. Anal. Appl. 328 (2007) 1108–1116.

[47] W. Walter, Ordinary Differential Equations, Springer, New York, 1998.

[48] Y. Yi, A generalized integral manifold theorem, J. Differential Equations 102 (1993) 153–187.