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COUNTING UNIFORMLY ATTRACTING SOLUTIONS OF NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract. Bounded uniform attractors and repellors are the natural nonautonomous analogues of autonomous stable and unstable equilibria. Unlike for equilibria, it is generally a difficult dynamical task to determine the number of uniformly attracting or repelling solutions for a given nonautonomous equation, even if the latter exhibits strong structural properties such as e.g. polynomial growth in space or periodicity in time. The present note highlights this aspect by proving that the number of uniform attractors is locally finite for several classes of equations, and by providing examples for which this number can be any $N \in \mathbb{N}$. These results and examples extend and complement recent work on nonautonomous differential equations.

1. **Introduction.** Stability and bifurcation theory of finite-dimensional ordinary differential equations

$$\dot{x} = F(x; \lambda)\,, \tag{1}$$

depending on a parameter λ, is a highly developed and to a large extent classical subject [1, 5, 6]. For the nonautonomous analogue of (1),

$$\dot{x} = f(t, x; \lambda)\,, \tag{2}$$

stability and especially bifurcations are by far less well understood; they are the subject of intense research [3, 4, 7–11, 15, 16]. Besides the enormous dynamical variety brought about by an explicit time-dependence, one patent difficulty inherent to (2) is that it is not at all obvious the bifurcations of which objects one should study. While classical bifurcation theory for (1) describes the change of stability as well as the creation and annihilation of equilibria, periodic and homoclinic orbits etc., equilibria and periodic orbits for instance are not generic for (2) if f depends aperiodically on t. To deal with these difficulties, uniformity in $t \in \mathbb{R}$ of some sort or another is typically assumed, or the transition of attractors is studied from a qualitative point of view only [11]. Stronger results are available for special cases, e.g., if some solutions can be computed explicitly [9, 10].

To bring forward internal attractor bifurcation analysis, *bounded uniform attractors* and *repellors* have been introduced in [2]. A bounded uniform attractor of

$$\dot{x} = f(t, x) \tag{3}$$

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is, by definition, a bounded solution which attracts all neighbouring solutions uniformly in $t \in \mathbb{R}$. Clearly, every uniform attractor is both forward and pull-back attracting, but in general the converse does not hold even in the simplest nonautonomous situations. Rather, uniform attraction is a significantly stronger property, and the results in [2] suggest that uniformly attracting solutions of (2) may be crucial for understanding local aspects of nonautonomous bifurcations. The purpose of this note is to complement these results by addressing the difficult question of determining the possible number of uniform attractors for a given class of equations and to provide several instructive examples; as in [2], the focus is on equations (3) which, albeit nonautonomous, exhibit some additional structure such as asymptotic autonomy, polynomial growth in x or periodicity in t.

2. **Uniform attractors and repellors.** This note studies differential equations (3) where $f : \mathbb{R}^2 \to \mathbb{R}$ as well as $\frac{\partial}{\partial x} f$ are continuous in (t, x); in addition it will be assumed throughout that $\sup_{\mathbb{R}\times K} |f| < \infty$ for every compact set $K \subset \mathbb{R}$, i.e., $f(t, \cdot)$ is uniformly bounded in t on every compact subset of $\mathbb{R}$. Given $(t_0, x_0) \in \mathbb{R}^2$ the initial value problem consisting of (3) together with $x(t_0) = x_0$ has a unique solution $t \mapsto \varphi(t; t_0, x_0)$ defined on some (possibly bounded) maximal open interval containing t_0. The following definition reflects the fact that a bounded solution of (3) can attract neighbouring solutions in different ways; for the sake of brevity the term *attractor* will be used instead of the accurate yet clumsy *attracting solution*.

Definition 1. Let $\mu : \mathbb{R} \to \mathbb{R}$ be a bounded solution of (3) and $(x_{0,\sigma})_{\sigma\in\mathbb{R}}$ a family of real numbers. Then μ is called

(i) a **forward attractor** if there exists $\delta > 0$ such that, for every $t_0 \in \mathbb{R}$,

$$|\varphi(t_0 + \tau; t_0, x_{0,t_0}) - \mu(t_0 + \tau)| \to 0 \qquad \text{as } \tau \to +\infty\,,$$

whenever $\|x_{0,\cdot} - \mu(\cdot)\|_\infty = \sup_{\sigma\in\mathbb{R}} |x_{0,\sigma} - \mu(\sigma)| < \delta$;

(ii) a **pullback attractor** if there exists $\delta > 0$ such that, for every $t \in \mathbb{R}$,

$$|\varphi(t; \sigma, x_{0,\sigma}) - \mu(t)| \to 0 \qquad \text{as } \sigma \to -\infty\,,$$

whenever $\|x_{0,\cdot} - \mu(\cdot)\|_\infty < \delta$;

(iii) a **uniform attractor** if there exists $\delta > 0$ such that

$$\|\varphi(\cdot + \tau; \cdot, x_{0,\cdot}) - \mu(\cdot + \tau)\|_\infty \to 0 \qquad \text{as } \tau \to +\infty\,,$$

provided that $\|x_{0,\cdot} - \mu(\cdot)\|_\infty < \delta$.

Moreover, μ is a **uniform repellor** if $t \mapsto \mu(-t)$ is a uniform attractor with t replaced by $-t$ in (3). Also, μ is referred to as a **global** forward, pullback, uniform attractor or uniform repellor if the respective property above holds for *every* $\delta > 0$.

Example 2. Equation (3) may have infinitely many uniform attractors, as can be seen for instance from

$$\dot{x} = -\tfrac{1}{2} e^t (1 + e^t)^{-1} x + x \sin\bigl(\pi x^2 (1 + e^t)\bigr)\,, \tag{4}$$

for which the functions μ_n and $-\mu_n$, with μ_n defined as

$$\mu_n : t \mapsto \sqrt{n}\,(1 + e^t)^{-1/2} \qquad (n \in \mathbb{N}_0)\,,$$

are uniform attractors whenever n is odd. Note that the right-hand side of (4) is real-analytic in x (and t) and that, for every compact set $K \subset \mathbb{R}$, the stripe $\mathbb{R} \times K$ contains only finitely many uniform attractors even though all attractors are joined at $+\infty$ in the sense that $\lim_{t\to+\infty}|\mu_n(t)-\mu_m(t)|=0$ for all n,m. If, however, the sine-function in (4) is replaced by the C^∞-function

$$H(z)=\begin{cases}0 & \text{if } z=0\,,\\ -e^{-\pi|z|^{-1}}\sin(\pi^2 z^{-1}) & \text{if } z\neq 0\,,\end{cases}\tag{5}$$

then the functions μ_n and $-\mu_n$, with μ_n given by

$$\mu_n: t\mapsto \tfrac{1}{\sqrt{n}}(1+e^t)^{-1/2}\qquad (n\in\mathbb{N})\,,$$

are uniform attractors for every odd n, and $\lim_{n\to+\infty}\mu_n(t)=0$ uniformly on $\mathbb{R}$; there are thus infinitely many uniform attractors contained in the stripe $\mathbb{R}\times[0,1]$.

In the autonomous case, that is for $f(t,x)\equiv F(x)$ not depending on t, all three notions of attraction coincide and every uniform attractor (repellor) μ is constant, $\mu(t)\equiv\mu_0$, with $F(\mu_0)=0$ and $(x-\mu_0)F(x)<0$ (>0) whenever $|x-\mu_0|>0$ is sufficiently small [2, Thm.9]. By analogy, and in view of Example 2, one might conjecture that if f is real-analytic in x for each t then, for every compact set $K\subset\mathbb{R}$, only finitely many uniform attractors and repellors are entirely contained in the stripe $\mathbb{R}\times K$. This, however, is not true in general, as evidenced by

Example 3. With H as in (5) and the parameter $0\le\kappa\le\frac{1}{3}$ consider the equation

$$\dot{x}=xH\big(\pi(x^2+e^{-t^2})\big)+\kappa xH\big(\pi(x^2+e^{-t^2})\big)^2=:f_\kappa(t,x)\,,\tag{6}$$

the right-hand side of which is real-analytic in x for each t. As will be explained below, (6) exhibits a sequence (μ_n) of uniform attractors with $\mu_1>\mu_2>\ldots>0$ if κ is chosen appropriately. Since several of the subsequent steps require for their justification elementary yet lengthy calculations, the argument will be outlined only to such an extent that the interested reader can easily fill in the details.

First define, for every $m\in\mathbb{N}_0$, the set

$$A_m=\big\{(t,x)\in\mathbb{R}^2: m<(x^2+e^{-t^2})^{-1}<m+1\big\}\,,$$

and observe that, for $x>0$, $f_\kappa(t,x)$ is positive (negative) if and only if $(t,x)\in A_m$ for some odd (even) m. Since $f_\kappa(t,1)<0$ for all t, the solution $\varphi(\cdot;t_0,x_0)$ exists for all $t\ge t_0$ if $0\le x_0\le 1$; similarly, if $0\le x_0\le\frac{1}{\sqrt{2}}$ then $\varphi(\cdot;t_0,x_0)$ exists for all $t\le t_0$ provided that $t_0\le-\sqrt{\log 2}$. Thus with $\rho=\varphi(0;-\sqrt{\log 2},\frac{1}{\sqrt{2}})>0$ the solution $\varphi(\cdot;0,x_0)$ of (6) exists for all t whenever $0\le x_0\le\rho$.

Next observe that, locally uniformly in x,

$$f_\kappa(t,x)\to xH(\pi x^2)+\kappa xH(\pi x^2)^2=:F_\kappa(x)\qquad\text{as } |t|\to\infty\,,$$

and also $\frac{\partial}{\partial x}f_\kappa(t,x)\to F'_\kappa(x)$. From this and a careful qualitative sketch of the sets A_m, it can be seen that, for every $0<x_0\le\rho$, the limit $\lim_{t\to\pm\infty}\varphi(t;0,x_0)$ exists and is in fact of the form $\frac{1}{\sqrt{m}}$ for some $m\in\mathbb{N}$. Here and throughout, usage of the symbol $\pm$ indicates that the respective expression, equation, etc. is to be read twice, once with the upper and once with the lower symbol(s) only. Let $M^\pm\in\mathbb{N}$ be such that $\lim_{t\to\pm\infty}\varphi(t;0,\rho)=\frac{1}{\sqrt{M^\pm}}$, and define

$$L_m^\pm=\big\{0<\xi<\rho:\lim\nolimits_{t\to\pm\infty}\varphi(t;0,\xi)=\tfrac{1}{\sqrt{m}}\big\}\qquad(m\ge M^\pm)\,.$$

It is easy to see that all these sets are non-empty (possibly one-point) intervals. Moreover, L_m^+ is open whenever m is odd. If $\xi\in L_m^+$ for some even m then

$$\sqrt{\tfrac{1}{\sqrt{m}}-e^{-t^2}}<\varphi(t;0,\xi)<\tfrac{1}{\sqrt{m}}$$

for all sufficiently large t. This together with the fact that $\frac{\partial}{\partial x} f_\kappa\big(t, \varphi(t;0,\xi)\big) > 0$ for all large t shows that L_m^+ cannot have positive length and hence is a singleton. Similarly, L_m^- is a singleton or an open interval depending on whether m is odd or even. Denoting the endpoints of $L_m^\pm$ by $l_m^\pm, l_{m-1}^\pm$ with $l_m^\pm \le l_{m-1}^\pm$ therefore

$$L_m^\pm = \begin{cases} \;]l_m^\pm, l_{m-1}^\pm[& \text{if } m \text{ is } \begin{matrix}\text{odd}\\ \text{even}\end{matrix} \;, \\ \{l_m^\pm\} = \{l_{m-1}^\pm\} & \text{if } m \text{ is } \begin{matrix}\text{even}\\ \text{odd}\end{matrix} \;. \end{cases}$$

If $x > 0$ then $\kappa \mapsto f_\kappa(t,x)$ is strictly increasing unless (t,x) belongs to the boundary of some set A_m. From this it follows that $\kappa \mapsto l_m^+(\kappa)$ is strictly decreasing for all $m \ge M^+$ and, analogously, $\kappa \mapsto l_m^-(\kappa)$ is strictly increasing for all $m \ge M^-$. Consequently, each set

$$K_m = \big\{\kappa \in [0, \tfrac{1}{3}] : l_m^-(\kappa) = l_k^+(\kappa) \text{ for some } k \ge M^+\big\} \qquad (m \ge M^-)\,,$$

is countable, and so is $K = \bigcup_{m \ge M^-} K_m$. Pick κ_0 from $[0, \frac{1}{3}]\backslash K$. For each $m \ge M^-$ there exists an odd number $k_m \ge M^+$ such that $l_m^-(\kappa_0) \in L_{k_m}^+$. Since the sequence $(k_m)_{m \ge M^+}$ is increasing and unbounded there exist odd numbers $m_1 < m_2 < \ldots$ such that $(k_{m_n})_{n\in\mathbb{N}}$ is strictly increasing. With these preparations define

$$\mu_n : t \mapsto \varphi\big(t; 0, l_{m_n}^-(\kappa_0)\big) \qquad (n \in \mathbb{N})\,.$$

For each odd $m \in \mathbb{N}$ there exist positive numbers T_m, δ_m, c_m such that

$$\tfrac{\partial}{\partial x} f_\kappa(t,y) \le -c_m \qquad \forall (t,y) : |t| \ge T_m, \big|y - \tfrac{1}{\sqrt{m}}\big| < \delta_m\,.$$

This together with the continuous dependence of $\varphi(t;0,\xi)$ upon ξ implies that μ_n is a uniform attractor for all $n \in \mathbb{N}$. Since $\varphi(t;0,\xi) > \varphi(t;0,\eta)$ for all t whenever $\xi > \eta$, it follows that $\mu_1 > \mu_2 > \ldots > 0$, and also that $\|\mu_n\|_\infty \to 0$ as $n \to \infty$.

Thus to guarantee local finiteness of the number of uniform attractors and repellors of (3) the class of admissible functions f has to be narrowed. In the next section asymptotically autonomous and time-periodic equations will be studied. Another important special case of (3) occurs if f is polynomial in x. Polynomial equations

$$\dot{x} = a_0(t) + a_1(t)x + \ldots + a_d(t)x^d\,, \tag{7}$$

with $d \in \mathbb{N}_0$ independent of t, and bounded continuous coefficients $a_0, a_1, \ldots, a_d$, have been studied extensively, not least for their connection with Hilbert's Sixteenth Problem [12–14, 17, 18]. In view of Example 3 it is tempting to formulate

Conjecture 4. *The total number of uniform attractors and repellors of* (7) *with* $d \in \mathbb{N}_0$ *and bounded continuous functions* $a_0, a_1, \ldots, a_d$ *is finite.*

In [2] this conjecture is verified (and d shown to be an upper bound on the total number) for $d \le 2$, and also for $d = 3$ if a_3 does not change its sign and $\int_0^\infty |a_3(t)|\,dt = \int_{-\infty}^0 |a_3(t)|\,dt = +\infty$. Although further special cases will be settled below, no overall proof of (or counterexample to) Conjecture 4 is known to the author. Note also that the stipulated uniformity is essential as e.g. every solution of (7) with $d = 1$ and $a_0 = 0$, $a_1(t) = -\mathrm{Arctan}\, t$ is a global *forward* attractor.

3. **Asymptotically autonomous and periodic equations.** Recall that (3) is termed (*two-sided*) *asymptotically autonomous* if for two functions $f_\pm : \mathbb{R} \to \mathbb{R}$

$$\lim\nolimits_{t \to \pm\infty} f(t,x) = f_\pm(x)$$

holds locally uniformly in x; additional regularity (e.g., Lipschitz continuity) is usually assumed for $f_\pm$ to ensure that the (autonomous) limiting equation

$$\dot{x} = f_\pm(x) \tag{8}$$

has unique local solutions. The following theorem generalises [2, Thm.17] and also has an immediate bearing on the counting problem.

Theorem 5. *Let* (3) *be asymptotically autonomous and assume that the solutions of* (8) *are locally unique. Also, let μ be a uniform attractor or repellor of* (3)*. Then the limit $\mu_\pm = \lim_{t\to\pm\infty}\mu(t)$ exists and is an isolated zero of $f_\pm$.*

Proof. By [2, Thm.13] the limit $\mu_\pm$ exists, and $f_\pm(\mu_\pm) = 0$. All that remains to be shown is that $\mu_\pm$ is an *isolated* zero of $f_\pm$. Since the argument for backward time is completely analogous, only the assertion about f_+ and μ_+ is proved here. To this end assume that $f_+(\mu_n) = 0$ for all n and some decreasing sequence (μ_n) with $\lim_{n\to+\infty}\mu_n = \mu_+$. The following argument shows that this assumption is incompatible with μ being a uniform attractor.

Given $\delta > 0$, pick $n \in \mathbb{N}$ such that $\mu_+ < \mu_n < \mu_+ + \frac{1}{2}\delta$, and let $\varepsilon = \frac{1}{3}(\mu_n - \mu_+)$. For $C > 0$, consider the autonomous initial value problem

$$\dot{y} = f_+(y) - C\,, \qquad y(0) = \mu_n\,. \tag{9}$$

Since $f_+(\mu_n) = 0$ and the solution $y = y(t)$ of (9) is locally unique, for every $L > 0$ a number $C = C_L > 0$ can be chosen so small that

$$\inf\{t \ge 0 : y(t) = \mu_n - \varepsilon\} > L\,.$$

Pick T_L large enough to ensure that both $|\mu(t) - \mu_+| < \varepsilon$ and

$$|f(t,x) - f_+(x)| < C_L \qquad \forall x : |x - \mu_n| \le \varepsilon$$

hold for all $t \ge T_L$. Since $|\mu_n - \mu(T_L)| < \delta$, the solution $\varphi(\cdot; T_L, \mu_n)$ tends to μ_+ as $t \to +\infty$. Hence the numbers

$$b = \inf\bigl\{t \ge T_L : \varphi(t; T_L, \mu_n) = \mu_n - \varepsilon\bigr\}$$

as well as

$$a = \sup\bigl\{T_L \le t < b : \varphi(t; T_L, \mu_n) = \mu_n\bigr\}$$

are both finite, and $T_L \le a < b$. Furthermore, the estimate $f(t,x) > f_+(x) - C_L$ for all $t \ge T_L$ and $x \in [\mu_n - \varepsilon, \mu_n]$ implies that $b - a \ge L$. Define now $(x_{0,\sigma})_{\sigma\in\mathbb{R}}$ as $x_{0,\sigma} = \mu_n$ if $\sigma = a$, and $x_{0,\sigma} = \mu(\sigma)$ otherwise. Then

$$\|x_{0,\cdot} - \mu(\cdot)\|_\infty = |\mu_n - \mu(a)| \le |\mu_n - \mu_+| + |\mu_+ - \mu(a)| < \delta\,,$$

but also, for all $0 \le \tau \le L$,

$$\begin{aligned}\|\varphi(\cdot+\tau;\cdot,x_{0,\cdot}) - \mu(\cdot+\tau)\|_\infty &= |\varphi(a+\tau; a, \mu_n) - \mu(a+\tau)| \\ &\ge |\mu_+ - \mu_n| - |\mu_n - \varphi(a+\tau; a,\mu_n)| - |\mu_+ - \mu(a+\tau)| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon\,.\end{aligned}$$

Since L was arbitrary, μ cannot be a uniform attractor. $\square$

Corollary 6. *Assume that* (3) *is asymptotically autonomous and $f_\pm$ is real-analytic. Then, for every compact set $K \subset \mathbb{R}$, the stripe $\mathbb{R}\times K$ contains only finitely many uniform attractors and repellors.*

Proof. If, under the stated assumptions, (3) has a uniform attractor or repellor then $f_\pm$ does not vanish identically and therefore, for every compact set $K \subset \mathbb{R}$, has only finitely many, say $N^\pm$, zeros in K. Denote by $\mu_\pm^{(1)} < \mu_\pm^{(2)} < \ldots < \mu_\pm^{(N^\pm)}$ all different zeros of $f_\pm$ in K. Also, let $\mu_1 < \ldots < \mu_N$ be any finite family of uniform attractors and repellors of (3). According to Theorem 5 there exist numbers $1 \le m_1^\pm \le \ldots \le m_N^\pm \le N^\pm$ such that

$$\lim_{t\to\pm\infty} \mu_i(t) = \mu_\pm^{(m_i^\pm)} \qquad (i = 1, \ldots, N).$$

If μ_i is a uniform attractor then $m_{i+1}^+ \ge m_i^+$ because solutions do not intersect over time. If $m_{i+1}^+ = m_i^+$ then μ_{i+1} must also be an attractor, and $m_{i+1}^- \ge m_i^- + 2$. This follows from the fact that for every solution x between μ_i and μ_{i+1} the non-empty set of all accumulation points of $\{x(t) : t \le 0\}$ must contain a zero of f_- between $\mu_-^{(m_i^-)}$ and $\mu_-^{(m_{i+1}^-)}$ that is different from both numbers. If, on the other hand, $m_{i+1}^+ > m_i^+$ then μ_{i+1} could be an attractor or a repellor and thus $m_{i+1}^- \ge m_i^- + 1$. In either case therefore $m_{i+1}^- + m_{i+1}^+ \ge m_i^- + m_i^+ + 2$. Completely analogous reasoning shows that the latter inequality also holds if μ_i is a uniform repellor. Since clearly $m_1^- + m_1^+ \ge 2$ it follows that $m_i^- + m_i^+ \ge 2i$ for all i. On the other hand, $m_i^- + m_i^+ \le N^- + N^+$ for all i, so that $2N \le N^- + N^+$. There are thus at most $\frac{1}{2}(N^- + N^+)$ uniform attractors and repellors entirely contained in $\mathbb{R} \times K$. □

Corollary 6 implies that Conjecture 4 does hold if (7) is asymptotically autonomous. In fact, the proof shows that d is an upper bound on the total number of uniform attractors and repellors in this case.

Remark 7. Corollary 6 does not require f to be, for each t, real-analytic in x. The analyticity of $f_\pm$, however, is essential as for instance the asymptotically autonomous equation (6) shows for which $f_\pm$ is merely C^∞.

Example 8. A standard condition ensuring that $\dot{y} = g(y)$ has locally unique solutions is that g be Lipschitz continuous. It is well-known that α-Hölder continuity for some $0 < \alpha < 1$ does not suffice for this purpose [19]. Correspondingly, Theorem 5 applies if $f_\pm$ is Lipschitz, but generally fails if it is only α-Hölder. For a concrete example, define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(t,x) = \begin{cases} 0 & \text{if } x = 0\,, \\ -xe^{-|x|^{-1}}\left(\sin^2(x^{-1}) + e^{-t^2}\right)^{\alpha/2} & \text{if } x \ne 0\,, \end{cases}$$

so that f is C^∞, and $f(t,x) \to f_\pm(x)$ locally uniformly as $t \to \pm\infty$, where

$$f_\pm(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ -xe^{-|x|^{-1}}|\sin(x^{-1})|^\alpha & \text{if } x \ne 0\,. \end{cases}$$

Note that $f_\pm$ is α-Hölder. The same argument as in [2, Exp.16] shows that the solution $\mu(t) \equiv 0$ is a uniform attractor, yet obviously not an isolated zero of $f_\pm$.

A second class of equations (3) for which the counting problem arises naturally consists of *periodic* equations. Assume from now on that f is T-periodic in t, i.e.,

$$f(t+T, x) = f(t,x)\,, \qquad \forall (t,x) \in \mathbb{R}^2\,. \tag{10}$$

with some $T > 0$. In this case, the long-time dynamics of (3) is governed by the *Poincaré map* $\Phi_T : x \mapsto \varphi(T; 0, x)$. The domain of Φ_T is some maximal open interval $I \subset \mathbb{R}$, and Φ_T is strictly increasing on I. As in the autonomous case, all three notions of attractions coincide for periodic equations. Moreover, every (forward, pullback, or uniform) attractor and uniform repellor μ is T-periodic, hence gives rise to, respectively, an attracting and a repelling fixed point $\mu(0)$ of Φ_T (see [2, 6] for details). The following observation also settles parts of Conjecture 4.

Theorem 9. *Let f satisfy* (10) *and assume that f is real-analytic in x for each $0 \le t < T$. Then for every compact set $K \subset \mathbb{R}$ the stripe $\mathbb{R} \times K$ contains only finitely many uniform attractors or repellors of* (3)*.*

Proof. Under the stated assumptions on f, [19, Thm.13.III] implies that Φ_T is real-analytic in its domain I. Assume that (μ_n) with $\mu_1 < \mu_2 < \ldots$ is a sequence of uniform attractors or repellors all of which are contained in $\mathbb{R} \times K$. Then $\mu : t \mapsto \sup_{n\in\mathbb{N}} \mu_n(t)$ is easily seen to be a T-periodic solution of (3) as well. Hence $\mu(0)$ is a fixed point of Φ_T, as is $\mu_n(0)$ for every n. Since μ is T-periodic, Φ_T is well-defined in some neighbourhood of $\mu(0)$. Hence $\mu(0)$ is an element of I. Thus the zeros of the real-analytic function $\Phi_T - id_I$ accumulate in I, and therefore $\Phi_T(x) \equiv x$. The latter, however, is impossible as it would imply that (3) does not have any uniform attractor or repellor at all. □

Corollary 10. *Let $a_0, a_1, \ldots, a_d$ be continuous and T-periodic ($d \in \mathbb{N}_0$). If $a_d(t) \neq 0$ for all t, then the total number of uniform attractors and repellors of* (7) *is finite.*

Proof. If $d \le 3$ then (7) has at most d uniform attractors and repellors [2, Thm.21]. Assume in turn that $d \ge 2$ and $a_d(t) \neq 0$ for all t. In this case, for every $t_0 \in [0, T]$ there exist positive numbers D_{t_0}, δ_{t_0} such that $|\nu(t)| \le D_{t_0}$ holds for all $|t - t_0| < \delta_0$ and every periodic solution ν of (7). Thus all uniform attractors and repellors of (7) are contained in $\mathbb{R} \times K$ with some compact interval $K \subset \mathbb{R}$. □

Under a definiteness assumption on a_d therefore the problem of counting uniform attractors and repellors of (7) with T-periodic coefficients arises naturally. It is well known that (7) may have many T-periodic solutions if $d \ge 4$. If a_d does change its sign then the situation is more intricate, and many T-periodic solutions may be found already for $d = 3$ (see [12, 13] for details). In [2] the relevance of the counting problem is highlighted further through several results about uniform attractors; in particular, an averaging type of argument is used to show that

$$\dot{x} = \varepsilon\big(a_0(t) + a_1(t)x + \ldots + a_d(t)x^d\big) \tag{11}$$

has, for all sufficiently small $\varepsilon > 0$, at most d uniform attractors and repellors provided that $\int_0^T a_d(t)\,dt \neq 0$. Also, if $\varepsilon \mapsto \mu_\varepsilon$ is a continuous parametrisation of periodic solutions of (11) then, uniformly in t, $\mu_\varepsilon(t) \to \mu_0$ as $\varepsilon \to 0$, where μ_0 denotes an equilibrium of the averaged (and hence autonomous) equation

$$\dot{x} = \varepsilon p(x) \quad \text{with} \quad Tp(x) = \int_0^T a_0(t)\,dt + x\int_0^T a_1(t)\,dt + \ldots + x^d\int_0^T a_d(t)\,dt\,. \tag{12}$$

As demonstrated below, the situation is more complicated in the resonant case, that is, for $\int_0^T a_d(t)\,dt = 0$; the following generalisation of [12, Lem.3.1] will be needed.

Lemma 11. *Let $I \subset \mathbb{R}$ be an open interval containing* 0*, and let $N \in \mathbb{N}$. Assume that for each $j = 1, \ldots, N+1$ the C^1-function $G_j : I \to \mathbb{R}$ has a finite non-zero limit*

$\lim_{x\to 0} x^{-g_j}G_j(x)$ for some $g_j \ge 0$. If the numbers $g_1,\ldots,g_{N+1}$ are all different, then there exist real numbers $\gamma_1,\ldots,\gamma_{N+1}$ such that the function

$$G = \gamma_1 G_1 + \ldots + \gamma_{N+1}G_{N+1}$$

has N zeros $x_1 < x_2 < \ldots < x_N$ in I with $G'(x_j) \neq 0$ for all $j = 1,\ldots,N$.

Proof. Without loss of generality assume that $g_1 > g_2 > \ldots > g_{N+1} \ge 0$, and also $\lim_{x\to 0} x^{-g_j}G_j(x) = 1$ for all j. Since $g_1 > 0$ there exists $\delta_1 > 0$ such that $G_1(x) > 0$ for all $0 < x \le \delta_1$. Let $H_1 = G_1$. Obviously, $\lim_{x\to 0} x^{-g_1}H_1(x) = 1$ and $H_1(x) > 0$ whenever $0 < x \le \delta_1$. Assume that positive numbers $\delta_1,\ldots,\delta_n$ have been found which satisfy $\delta_n < \frac{1}{2}\delta_{n-1} < \frac{1}{4}\delta_{n-2} < \ldots < 2^{1-n}\delta_1$, and that a linear combination H_n of $G_1,\ldots,G_n$ has been constructed with $\lim_{x\to 0} x^{-g_n}H_n(x) = 1$ and $H_n(x) > 0$ for all $0 < x \le \delta_n$, but also, for all $k = 1,\ldots,n-1$,

$$(-1)^k H_n(x) > 0 \qquad \forall x \in [\tfrac{1}{2}\delta_{n-k}, \delta_{n-k}]\,. \tag{13}$$

Choose $\eta_{n+1} > 0$ sufficiently small to ensure that $|H_n(x)| > 2\eta_{n+1}|G_{n+1}(x)|$ for all $x \in \bigcup_{k=0}^{n-1}[\frac{1}{2}\delta_{n-k}, \delta_{n-k}]$, and let

$$H_{n+1} = -\tfrac{1}{\eta_{n+1}}H_n + G_{n+1}\,. \tag{14}$$

It is easy to check that H_{n+1} thus defined satisfies (13) with n replaced by $n+1$, for all $k = 1,\ldots,n$. Furthermore, since $g_n > g_{n+1}$,

$$x^{-g_{n+1}}H_{n+1}(x) = -\tfrac{1}{\eta_{n+1}}x^{-g_n}H_n(x)x^{g_n-g_{n+1}} + x^{-g_{n+1}}G_{n+1}(x) \to 1 \qquad \text{as } x \to 0\,,$$

so that $0 < \delta_{n+1} < \frac{1}{2}\delta_n$ can be found with $H_{n+1}(x) > 0$ whenever $0 < x \le \delta_{n+1}$.

Carrying out N steps of (14) yields a linear combination H_{N+1} of $G_1,\ldots,G_{N+1}$ with $H_{N+1}(x) > 0$ whenever $0 < x \le \delta_{N+1}$, and, for all $j = 1,\ldots,N$,

$$(-1)^j H_{N+1}(x) > 0 \qquad \forall x \in [\tfrac{1}{2}\delta_{N+1-j}, \delta_{N+1-j}]\,.$$

Thus for each $j = 1,\ldots,N$ there exists $\overline{x}_j$ between δ_{N+2-j} and $\frac{1}{2}\delta_{N+1-j}$ with $H_{N+1}(\overline{x}_j) = 0$, hence H_{N+1} has N different zeros in the interval $[\delta_{N+1}, \frac{1}{2}\delta_1] \subset I$. To provide *simple* zeros assume without loss of generality that each $\overline{x}_j$ is the supremum or infimum of $H_{N+1}^{-1}(\{0\}) \cap [\delta_{N+2-j}, \frac{1}{2}\delta_{N+1-j}]$, depending on whether j is odd or even, and consider the auxiliary function $F(x,\eta) = H_{N+1}(x) + \eta G_1(x)$; note that $F(\overline{x}_j, 0) = 0$ and $\frac{\partial}{\partial\eta}F(\overline{x}_j, 0) = G_1(\overline{x}_j) > 0$. Thus for each $j = 1,\ldots,N$ there exists an open interval $I_j \subset [\delta_{N+2-j}, \frac{1}{2}\delta_{N+1-j}]$ containing $\overline{x}_j$, and a C^1-function $h_j : I_j \to \mathbb{R}$ with $h_j(\overline{x}_j) = 0$ and $F\big(x, h_j(x)\big) = 0$ for all $x \in I_j$. Note that $h_j(x) > 0$ for all $x \in I_j$ with $(-1)^{j+1}(x - \overline{x}_j) > 0$. Consequently, the image $h_j(I_j)$ is a non-degenerate interval containing $[0, \eta_j]$ for some $\eta_j > 0$, and the set $C_j = \{h_j(x) : x \in I_j, h_j'(x) = 0\}$ of critical values has measure zero. Pick $\eta_0 > 0$ from $\bigcap_{j=1}^N \big(h_j(I_j)\backslash C_j\big)$. For every $j = 1,\ldots,N$ there exists $x_j \in I_j$ such that $h_j(x_j) = \eta_0$ yet $(-1)^{j+1}h_j'(x_j) > 0$. Hence $F(x_j, \eta_0) = 0$ yet

$$\begin{aligned}
(-1)^j\big(H_{N+1}'(x_j) + \eta_0 G_j'(x_j)\big) &= (-1)^j \tfrac{\partial}{\partial x}F(x_j, \eta_0) \\
&= (-1)^{j+1}h_j'(x_j)G_1(x_j) \\
&> 0\,.
\end{aligned}$$

Thus each x_j is a simple zero of $G := H_{N+1} + \eta_0 G_1$ with $(-1)^j G'(x_j) > 0$. $\square$

Lemma 11 is instrumental in establishing the following generalisation of [2, Cor.30].

Theorem 12. *Assume that $d \geq 3$ and $a_d \neq 0$ is continuous and T-periodic with $\int_0^T a_d(t)\,dt = 0$. Then, given $N \in \mathbb{N}$, there exists a continuous T-periodic function a_2, satisfying both $\max(|a_2(t)|, |a_d(t)|) > 0$ for all t and $\int_0^T a_2(t)\,dt > 0$, such that*

$$\dot{x} = \varepsilon\bigl(-1 + a_2(t)x^2 + a_d(t)x^d\bigr) \tag{15}$$

has N uniform attractors whenever $\varepsilon > 0$ is sufficiently small.

Proof. For notational convenience, in (15) replace ε by ε^{d-1}. Setting $y = \varepsilon x$ transforms (15) into

$$\dot{y} = -\varepsilon^d + \varepsilon^{d-2}a_2(t)y^2 + a_d(t)y^d\,. \tag{16}$$

Let $A_d(t) = (1-d)\int_0^t a_d(s)\,ds$, so that in particular $A_d(0) = A_d(T) = 0$; also define $C_d = \bigl(\max_{0\leq t\leq T}|A_d(t)|\bigr)^{-\frac{1}{d-1}} > 0$. For $|z| < C_d$ and ε sufficiently small, and with

$$Q_{a_2}(z) = \int_0^T a_2(t)\bigl(1 + A_d(t)z^{d-1}\bigr)^{\frac{d-2}{d-1}}\,dt \qquad (|z| < C_d)\,, \tag{17}$$

the Poincaré map associated with (16) can be written in the form

$$\Phi_T(z) = z + \varepsilon^{d-2}z^2Q_{a_2}(z) + \varepsilon^{d-1}S(z,\varepsilon)\,; \tag{18}$$

here $S(z,\varepsilon)$ is real-analytic in z (and ε) and converges uniformly as $\varepsilon \to 0$. Assume that $z_0 > 0$ is a zero of Q_{a_2} with $Q'_{a_2}(z_0) < 0$. Since $\frac{\partial}{\partial z}\bigl(z^2Q_{a_2}(z) + \varepsilon S(z,\varepsilon)\bigr)\big|_{(z_0,0)} = z_0^2Q'_{a_2}(z_0) < 0$, for all sufficiently small $\varepsilon > 0$ there exists z_ε near z_0 such that $z_\varepsilon^2Q_{a_2}(z_\varepsilon) + \varepsilon S(z_\varepsilon,\varepsilon) = 0$, hence $\Phi_T(z_\varepsilon) = z_\varepsilon$, and

$$\begin{aligned} 0 \leq \Phi'_T(z_\varepsilon) &= 1 + \varepsilon^{d-2}\bigl(2z_\varepsilon Q_{a_2}(z_\varepsilon) + z_\varepsilon^2Q'_{a_2}(z_\varepsilon)\bigr) + \varepsilon^{d-1}\tfrac{\partial}{\partial z}S(z_\varepsilon,\varepsilon) \\ &< 1\,. \end{aligned}$$

By [2, Thm.23] the solution $\varphi(\cdot;0,z_\varepsilon)$ is a uniform attractor of (16). Thus the proof will essentially be complete once a function a_2 has been specified in such a way that Q_{a_2} has N simple zeros in $]0, C_d[$ with negative slope. To this end assume without loss of generality that $a_d(0) = 0$ and $a_d(t) \neq 0$ for all $0 < t < \delta$ with $\delta = \inf\{0 < t \leq T : a_d(t) = 0\} > 0$. Thus A_d is not constant on the interval $[0,\delta]$. For each $j = 1,\ldots,2N+1$ choose a continuous function $a_2^{(j)}$ with $a_2^{(j)}(0) = 0$ and $a_2^{(j)}(t) = 0$ for all $\delta \leq t \leq T$ such that

$$\forall k = 0,\ldots,j-1: \ \int_0^T a_2^{(j)}(t)A_d^k(t)\,dt = 0\,, \quad \text{yet} \quad \int_0^T a_2^{(j)}(t)A_d^j(t)\,dt = 1\,.$$

Such a choice is possible because A_d is continuous and not constant. Note that Q_{a_2} depends linearly upon a_2, and, for each $j = 1,\ldots,2N+1$,

$$\begin{aligned} Q_{a_2^{(j)}}(z) &= \sum\nolimits_{k=0}^{\infty} \begin{pmatrix} \frac{d-2}{d-1} \\ k \end{pmatrix} z^{k(d-1)} \int_0^T a_2^{(j)}(t)A_d^k(t)\,dt \\ &= z^{j(d-1)} \begin{pmatrix} \frac{d-2}{d-1} \\ j \end{pmatrix} + \mathcal{O}\bigl(z^{(j+1)(d-1)}\bigr)\,. \end{aligned}$$

Thus Lemma 11 applies with $G_j = Q_{a_2^{(j)}}$ and $g_j = j(d-1)$, yielding a continuous function a_2 with $a_2(0) = 0$ and $a_2(t) = 0$ for all $\delta \leq t \leq T$ such that $Q_{a_2} = G$ has $2N$ simple zeros in $]0, C_d[$ of which N have negative and N have positive slope. To conclude the proof, replace a_2 by $-a_2$, if necessary, to ensure that $\int_0^T a_2(t)\,dt \geq 0$.

For all sufficiently small $\rho > 0$, $Q_{a_2+\rho}$ also has N single zeros with negative slope. Moreover, $\int_0^T (a_2(t)+\rho)\,dt > 0$ and $\max(|a_2(t)|, |a_d(t)|) > 0$ for every $0 \le t \le T$. Thus, replacing a_2 by $a_2 + \rho$ with small positive ρ and extending it T-periodically finally yields a function that has all properties referred to in the theorem. $\square$

Remark 13. (i) Under the conditions of Theorem 12 the averaged equation (12) associated with (15) has exactly one uniform attractor $\mu_0 = -\big(\frac{1}{T}\int_0^T a_2(t)\,dt\big)^{-1/2}$ and one uniform repellor $-\mu_0$. In stark contrast to the non-resonant case, however, $\lim_{\varepsilon\to 0}\mu_\varepsilon(t) = 0$ holds uniformly in t for every uniform attractor μ_ε of (15). As $\varepsilon \to 0$, therefore, the latter equation exhibits what appears to be an intricate, genuinely nonautonomous bifurcation.

(ii) In the context of Theorem 12 it is natural to ask for the exact number of uniform attractors and repellors. Obviously, without additional hypotheses the perturbational nature of (18) rules out any general statement near the endpoints of the interval $]-C_d, C_d[$. Even for a compact subinterval of the latter, however, it will in general be difficult to find viable conditions guaranteeing exactly a given number of attractors and repellors. For concrete equations, obviously the situation may be much simpler. For a concrete example consider the special case $d = 3$, $T = 2\pi$, and let $a_3(t) = 4\sin t$, hence $A_3(t) = -16\sin^2(\frac{1}{2}t)$ and $C_3 = \frac{1}{4}$. With $a_2^{(j)} = \cos jt$ an evaluation of (17) yields, for every $j \in \mathbb{N}_0$,

$$(2j-1)z^{-2j}Q_{a_2^{(j)}}(z) = (-1)^{j+1}2\pi\binom{2j}{j}\big(1 + (8j-4)z^2 + \mathcal{O}(z^4)\big) \qquad \big(|z| < \tfrac{1}{4}\big)\,.$$

In fact, $z^{-2j}Q_{a_2^{(j)}}(z)$ is, for each $j \ge 1$, a smooth strictly convex function which in $[0, \frac{1}{4}]$ vanishes only at $z = 0$. With this additional structure it is not hard to show that, just as in the (hypothetical) monomial case $Q_{a_2^{(j)}}(z) = z^{2j}$, numbers γ_j can be found so as to make $\sum_j \gamma_j Q_{a_2^{(j)}} = Q_{\sum_j \gamma_j a_2^{(j)}}$ exhibit any given number of zeros.

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