

## INVARIANT MEASURES FOR GENERAL INDUCED MAPS AND TOWERS

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(Communicated by C. Liverani)

**ABSTRACT.** Absolutely continuous invariant measures (acims) for general induced transformations are shown to be related, in a natural way, to popular tower constructions regardless of any particulars of the latter. When combined with (an appropriate generalization of) the known integrability criterion for the existence of such acims, this leads to necessary and sufficient conditions under which acims can be lifted to, or projected from, nonsingular extensions.

**1. Introduction.** One basic idea in ergodic theory, of great importance both for abstract considerations and for the analysis of specific examples, is to study a dynamical system  $\mathfrak{S}$  acting on a space  $X$  by means of some closely related auxiliary system. A classical example [9] in this regard is the system  $\mathfrak{S}_Y$  obtained by passing to the *first return* (or *induced*) map on a suitable subset  $Y$  of  $X$ . In this case, the new system faithfully reflects many relevant properties of  $\mathfrak{S}$ , and hence establishing these properties for  $\mathfrak{S}_Y$  often is equivalent to, yet simpler than proving them for  $\mathfrak{S}$  directly — see subsequent sections for precise statements. A more flexible variant of first return maps, a *general induced system*  $\mathfrak{S}^\tau$  allows for an *inducing time*  $\tau$  more general than the first return time. General induced systems have become an extensively used tool in measurable dynamics. While they also act on appropriate subsets  $Y$  of  $X$ , a different, equally fundamental type of auxiliary construction enlarges rather than reduces the space  $X$ , resulting in an *extension*  $\mathfrak{S}^*$  of  $\mathfrak{S}$  that acts on a different set  $X^*$  but projects onto  $X$ .

The present article focuses on questions regarding absolutely continuous invariant measures (*acims*) for nonsingular systems. While (under mild assumptions) there is a one-to-one correspondence between the  $\sigma$ -finite acims for  $\mathfrak{S}$  and those for  $\mathfrak{S}_Y$ , the situation is more complicated for general induced systems and extensions, that is, for  $\mathfrak{S}^\tau$  and  $\mathfrak{S}^*$ . In either case, a  $\sigma$ -finite acim of the auxiliary system automatically yields an acim for  $\mathfrak{S}$ , but one needs to check separately whether that acim is  $\sigma$ -finite.

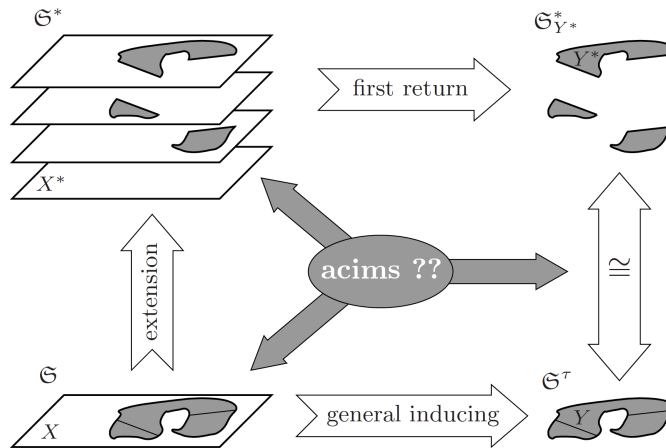
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2010 *Mathematics Subject Classification.* Primary 28D05, 37A25, 37C30.

*Key words and phrases.* Induced transformation, jump transformation, Markov extension, Hofbauer tower, Young tower, infinite ergodic theory.

Conversely, given a  $\sigma$ -finite acim for  $\mathfrak{S}$ , it is a nontrivial task to decide whether this acim can be obtained via general inducing or extension.

The resemblance between the constructions of  $\mathfrak{S}^\tau$  and  $\mathfrak{S}^*$ , to be recalled in detail in Section 2 below, is not a coincidence: *Very often general induced systems  $\mathfrak{S}^\tau$  correspond to first return systems  $\mathfrak{S}_{Y^*}^*$  in extensions.* This basic correspondence principle, to be made precise in the next section also, has been observed and used by many authors (see, in particular, [3, 4]). However, the principle has rarely been made explicit, and even when it has, this has so far been done solely in the context of fairly specific (tower) constructions. It seems worthwhile to isolate the essence of this correspondence, and to provide results that are general enough to apply directly to a wide variety of concrete constructions. The purpose of the present paper, therefore, is to fully capture this correspondence principle within the abstract framework of nonsingular ergodic theory.



Motivated by [3] and [4] the first main result, Theorem 3.1 below, clarifies the natural relation between the acims of  $\mathfrak{S}^\tau$  and the acims of  $\mathfrak{S}^*$ . Several generalizations of classical facts about induced maps which are of independent interest are required for the proof of Theorem 3.1. Aided by this theorem, it is possible to improve a key result of [18] about general induced systems. This in turn leads to the note's second main result, Theorem 3.3 below. The latter can be utilized in the context of many different types of extensions and yields, for instance, criteria for the liftability of acims to towers (Corollary 3.5) and for projections of acims from towers to be  $\sigma$ -finite (Corollary 3.8). Importantly, these criteria do not depend on any further particulars of the tower construction.

**2. Induced maps and towers.** With regard to the statements, explanations and proofs of the main results in subsequent sections, this preparatory section briefly reviews all the relevant aspects of (general) inducing and extensions.

**Nonsingular and measure preserving systems. Recurrence.** The appropriate basic notion for this article is that of a *nonsingular transformation*  $T$  on a measure space  $(X, \mathcal{A}, \lambda)$ , meaning that  $T : X \rightarrow X$  is a measurable map (not necessarily invertible) for which the image measure  $T\lambda := \lambda \circ T^{-1}$  is absolutely continuous w.r.t.  $\lambda$ , in symbols  $T\lambda \ll \lambda$ . While many interesting dynamical systems first present themselves in the form of a *nonsingular system*  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$ ,

one usually aims at equipping them with an *invariant* measure which is absolutely continuous w.r.t.  $\lambda$ , i.e., one aims at finding  $\mu \ll \lambda$  such that  $T\mu = \mu$ . In this situation, both  $T$  and the (special nonsingular) system  $(X, \mathcal{A}, \mu, T)$  are referred to as *measure preserving* (*mp*, for short).

Given  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$  and any set  $Y \in \mathcal{A}^+ := \{A \in \mathcal{A} : \lambda(A) > 0\}$ , the *first entrance time*  $\varphi_Y(x) := \inf\{n \geq 1 : T^n x \in Y\}$  of  $Y$  defines a measurable function  $\varphi_Y : X \rightarrow \overline{\mathbb{N}} := \{1, 2, \dots, \infty\}$ , with the usual convention that  $\inf \emptyset := \infty$ . When restricted to  $Y$ , the function  $\varphi_Y$  is referred to as the *first return time* of  $Y$ . If  $\varphi_Y < \infty$   $\lambda$ -a.e. on  $Y$ , that is, if  $Y \subseteq \bigcup_{n \geq 1} T^{-n}Y \bmod \lambda$ , then  $Y$  is a *recurrent set*; it is a *sweep-out set* if  $\bigcup_{n \geq 0} T^{-n}Y = X \bmod \lambda$ . Call  $\mathfrak{S}$  *conservative* if every  $A \in \mathcal{A}^+$  is recurrent, and *ergodic* if  $\lambda(A) = 0$  or  $\lambda(A^c) = 0$  holds whenever  $A \in \mathcal{A}$  is  $T$ -invariant, i.e., whenever  $T^{-1}A = A \bmod \lambda$ . For each recurrent  $Y$ , the smallest invariant set containing  $Y \bmod \lambda$  is  $Y_\infty := \bigcap_{n \geq 0} \bigcup_{j \geq n} T^{-j}Y$ ; note that  $Y_\infty = \bigcup_{n \geq 0} T^{-n}Y \bmod \lambda$ .

**First return maps.** Every recurrent set  $Y$  comes with a *first return* (or *induced*) map  $T_Y : Y \rightarrow Y$ , defined as  $T_Y x := T^{\varphi_Y(x)} x$  whenever  $\varphi_Y(x) < \infty$ , and  $T_Y x := x$  otherwise, which is nonsingular on  $(Y, \mathcal{A} \cap Y, \lambda|_Y)$ ,<sup>1</sup> thus defining a new system  $\mathfrak{S}_Y$ . The great importance of this classical construction is due to the fact that the two nonsingular systems  $\mathfrak{S}_Y$  and  $\mathfrak{S}|_{Y_\infty} := (Y_\infty, \mathcal{A} \cap Y_\infty, \lambda|_{Y_\infty}, T|_{Y_\infty})$  are intimately related. (The following statements are all contained in [1, Sec.1.5].) For instance,

$$\mathfrak{S}_Y \text{ is ergodic} \iff \mathfrak{S}|_{Y_\infty} \text{ is ergodic.} \quad (1)$$

Moreover, there is a well-known correspondence between absolutely continuous invariant measures (abbreviated henceforth as *acim*) associated with  $\mathfrak{S}_Y$  and  $\mathfrak{S}|_{Y_\infty}$ , respectively:

$$\begin{aligned} &\text{If } \nu \ll \lambda|_Y \text{ is } T_Y\text{-invariant, then } \nu = \mu|_Y \text{ for the } T\text{-invariant} \\ &\text{measure } \mu \ll \lambda|_{Y_\infty} \text{ given by } \mu(A) := \sum_{n \geq 0} \nu(Y \cap \{\varphi_Y > n\} \cap T^{-n}A); \end{aligned} \quad (2)$$

and a partial converse of (2) reads:

$$\begin{aligned} &\text{If } \mu \ll \lambda \text{ is } T\text{-invariant, and } \mu(Y) < \infty, \\ &\text{then } \nu := \mu|_Y \ll \lambda|_Y \text{ is } T_Y\text{-invariant.} \end{aligned} \quad (3)$$

This correspondence can be used in either direction to find acims. In the situation of (3),  $\mu$  is clearly  $\sigma$ -finite on  $Y_\infty$ . Notice also that

$$\text{in (2) the measure } \mu \text{ is } \sigma\text{-finite iff } \nu \text{ is } \sigma\text{-finite.} \quad (4)$$

Indeed,  $\nu$  is  $\sigma$ -finite whenever  $\mu$  is, and for the converse assume that  $Y = \bigcup_{j \geq 1} Z_j^0$  with  $\nu(Z_j^0) < \infty$ , note that  $\mu(Y_\infty^c) = 0$ , and cover  $Y_\infty \setminus Y \bmod \lambda$  by the sets  $Z_j^n := (Y_\infty \setminus Y) \cap \{\varphi_Y = n\} \cap T^{-n}Z_j^0$  which satisfy  $\mu(Z_j^n) \leq \nu(Z_j^0) < \infty$  for all  $j, n \geq 1$ .

**General induced transformations.** The concept of first return maps for a nonsingular system  $(X, \mathcal{A}, \lambda, T)$  has a far-reaching generalisation which allows for other accelerated versions of  $T$ , and thus provides a very flexible method of constructing convenient auxiliary transformations associated with  $T$ . As in [18], call a measurable function  $\tau : X \rightarrow \overline{\mathbb{N}}$  a (*general*) *inducing time for  $T$  on  $Y \in \mathcal{A}^+$*  if it is finite

<sup>1</sup>For the sake of brevity, for any  $A \in \mathcal{A}$  denote by  $\lambda|_A$  the restriction of  $\lambda$  to  $\mathcal{A} \cap A$ ; also say that  $\lambda$  has some property *on*  $A$  if  $\lambda|_A$  has that property. For example,  $\lambda = \lambda'$  on  $A$  means that  $\lambda|_A = \lambda'|_A$ , etc. Similarly, if  $A \subset T^{-1}A$  then the system  $(X, \mathcal{A}, \lambda, T)$  is said to have a certain property, e.g. ergodicity, *on*  $A$  whenever  $(A, \mathcal{A} \cap A, \lambda|_A, T|_A)$  has that property.

a.e. on  $Y$ , with  $T^\tau x := T^{\tau(x)}x \in Y$  for a.e.  $x \in Y$ . (Note that  $Y$  necessarily is recurrent in this case.) The map  $T^\tau : Y \rightarrow Y$  then is a nonsingular transformation on  $(Y, \mathcal{A} \cap Y, \lambda|_Y)$ , referred to as the transformation which  $T$  and  $\tau$  induce on  $Y$ . The new system thus obtained will be denoted  $\mathfrak{S}^\tau$ . With this, first return maps simply constitute the special case  $\tau = \varphi_Y$  with any recurrent set  $Y$ , that is,  $\mathfrak{S}_Y = \mathfrak{S}^{\varphi_Y}$ .

Section 5 below reviews basic features of general induced systems, and also records several new observations which are of independent interest. As a rule,  $\mathfrak{S}$  still inherits many important properties from  $\mathfrak{S}^\tau$ , but the opposite direction is more difficult than in the classical case of  $\mathfrak{S}_Y$  outlined earlier. For example, while one implication of (1) generalizes (see [14, Sec.1]) to

$$\mathfrak{S}^\tau \text{ is ergodic} \implies \mathfrak{S}|_{Y_\infty} \text{ is ergodic,}$$

the converse implication breaks down for general  $\tau$ . Similarly, an acim for  $\mathfrak{S}^\tau$  always yields an acim for  $\mathfrak{S}$  via the following canonical construction which generalizes that of (2): Given any general inducing time  $\tau$  on  $Y$  and any measure  $\nu \ll \lambda|_Y$  on  $(Y, \mathcal{A} \cap Y)$ , define a new measure  $\tau \times_T \nu$  on  $(X, \mathcal{A})$  according to

$$\tau \times_T \nu(A) := \sum_{n \geq 0} \nu(Y \cap \{\tau > n\} \cap T^{-n}A), \quad \forall A \in \mathcal{A}.$$

Then,  $\tau \times_T \nu \ll \lambda|_{Y_\infty}$ , and (again according to [14, Sec.1])

$$\text{if } \nu \text{ is } T^\tau\text{-invariant, then } \mu := \tau \times_T \nu \text{ is } T\text{-invariant,} \quad (5)$$

but  $\mu$  need not be  $\sigma$ -finite even if  $\nu$  is. In contrast to the converse (3) in the case of first return maps, there is no simple, let alone canonical way of turning a  $T$ -invariant  $\mu$  into some  $T^\tau$ -invariant  $\nu$  that satisfies  $\tau \times_T \nu = \mu$ . (For ergodic finite invariant measures  $\mu$  this matter has been discussed in [18], where necessary and sufficient conditions for the existence of  $\nu$  are given.)

**Nonsingular extensions.** Passing from  $\mathfrak{S}$  to an induced system  $\mathfrak{S}^\tau$  means combining several iterations of  $T$  into into a single iteration of a new map, and possibly also passing to a smaller space. However, studying finer properties of  $\mathfrak{S}$  sometimes is facilitated by going in the opposite direction and constructing a larger system  $\mathfrak{S}^*$  that provides enough space for the unfolding of complicated bits by keeping orbits of different types separated. Specifically, a *nonsingular extension* of  $\mathfrak{S}$  is a nonsingular system  $\mathfrak{S}^* = (X^*, \mathcal{A}^*, \lambda^*, T^*)$  together with a *nonsingular factor map*  $\pi : X^* \rightarrow X$ , that is,  $\pi$  is a measurable map with  $\pi\lambda^*$  equivalent to  $\lambda$ , such that  $\pi \circ T^* = T \circ \pi$  holds  $\lambda^*$ -a.e. on  $X^*$ . Again, such auxiliary systems  $\mathfrak{S}^*$  are useful because their ergodic properties are often passed on to  $\mathfrak{S}$ . For example, it is easily seen that

$$\mathfrak{S}^* \text{ is ergodic} \implies \mathfrak{S} \text{ is ergodic.}$$

The reverse implication, however, trivially fails. Regarding measures, since  $\pi$  is nonsingular, one has  $\pi\mu^* \ll \lambda$  whenever  $\mu^* \ll \lambda^*$ . In this case, it is immediate that

$$\text{if } \mu^* \text{ is } T^*\text{-invariant, then } \mu := \pi\mu^* \text{ is } T\text{-invariant,} \quad (6)$$

but  $\mu$  need not be  $\sigma$ -finite even if  $\mu^*$  is. Again the converse presents difficulties, and a frequently encountered question is whether or not a  $T$ -invariant measure  $\mu \ll \lambda$  on  $X$  can be *lifted* to an acim on  $X^*$ , that is, whether there is some  $T^*$ -invariant  $\mu^* \ll \lambda^*$  for which  $\pi\mu^* = \mu$ .

**Relation between general induced maps and towers.** The discussion of  $\mathfrak{S}^\tau$  and  $\mathfrak{S}^*$  above highlights some analogies between these concepts that were mentioned informally already in the introduction. Why would one want to more formally study

the relation between the two concepts? A most compelling reason may be seen in the fact that quite often either concept is applied in situations of the following type: Let  $\mathfrak{S}^*$  be a nonsingular extension of  $\mathfrak{S}$ , and assume  $Y^* \in \mathcal{A}^*$  is such that, for some  $Y \in \mathcal{A}$ , the restricted factor map  $\pi|_{Y^*} : Y^* \rightarrow Y$  is invertible as a nonsingular map. Then  $Y^*$  is a copy of  $Y$  embedded in  $X^*$ . The term *tower over*  $\mathfrak{S}$  is often used for particular types of nonsingular extensions  $\mathfrak{S}^*$  where  $X^*$  is a countable union of embedded subsets  $Y^*$  as above. Many specific constructions are of this type, for example the *canonical Markov extensions* of [10, 11] and variants thereof, but also the *Young towers* in [15, 16].

Now, the abstract considerations of this article apply to every extension  $\mathfrak{S}^*$  with a nontrivial embedded set  $Y^*$  that is recurrent. Fix any such  $Y^*$ , and denote by  $Y_\infty^*$  the smallest invariant set containing  $Y^*$ . According to the discussion earlier in this section, passing from  $\mathfrak{S}^*|_{Y_\infty^*}$  to the first return system  $\mathfrak{S}_{Y^*}^*$  preserves crucial features of that part of the system  $\mathfrak{S}^*$ . Observe then that this first return system in the extension is *isomorphic* as a nonsingular system to a general induced system  $\mathfrak{S}^\tau$  for  $\mathfrak{S}$ , that is, there is an invertible nonsingular factor map from one system onto the other. Indeed, the function  $\tau : Y \rightarrow \bar{\mathbb{N}}$  given by  $\tau := \varphi_{Y^*}^* \circ \pi|_{Y^*}^{-1}$ , with  $\varphi_{Y^*}^*$  denoting the first entrance time of  $Y^*$  under  $T^*$ , is easily seen to be an inducing time for  $T$ , and

$$T^\tau = \pi \circ T_{Y^*}^* \circ \pi|_{Y^*}^{-1} \quad \lambda\text{-a.e. on } Y.$$

Thus, first return systems  $\mathfrak{S}_{Y^*}^*$  on recurrent embedded sets  $Y^*$  in an extension  $\mathfrak{S}^*$  always correspond to generalized induced systems  $\mathfrak{S}^\tau$  for  $\mathfrak{S}$ . Conversely, as pointed out in [18], every  $\mathfrak{S}^\tau$  is isomorphic (as a nonsingular system) to a first return system  $\mathfrak{S}_{Y^*}^*$  in a suitable extension  $\mathfrak{S}^*$  of  $\mathfrak{S}$ . Any extension  $\mathfrak{S}^*$  related in this way to a given induced system  $\mathfrak{S}^\tau$  will be called  *$\tau$ -trivialising*, as it allows to represent the general induced transformation  $T^\tau$  as a first return map. In view of the preceding discussion, this definition is crucial for all that follows.

**Definition 2.1.** Let  $\mathfrak{S}^* = (X^*, \mathcal{A}^*, \lambda^*, T^*)$  be a nonsingular extension of  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$  with factor map  $\pi : X^* \rightarrow X$ , and  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ . The extension  $\mathfrak{S}^*$  is  *$\tau$ -trivialising*, with  *$\tau$ -base*  $Y^* \in \mathcal{A}^*$ , provided that  $\pi|_{Y^*} : Y^* \rightarrow Y$  is invertible (as a nonsingular map) and, with  $\varphi_{Y^*}^* : X^* \rightarrow \bar{\mathbb{N}}$  denoting the first entrance time of  $Y^*$  under  $T^*$ ,

$$\tau \circ \pi = \varphi_{Y^*}^* \quad \lambda^*\text{-a.e. on } Y^*. \tag{7}$$

**Remark 2.2.** Implicit in this definition is the requirement that  $Y^*$  be a recurrent set in the extension  $\mathfrak{S}^*$ .

The organisation of the remainder of this article is as follows. The main results are stated in Section 3. For the reader's convenience, Section 4 provides a worked-out classical example illustrating the notion of  $\tau$ -trivialising extension and also contains further comments regarding this concept. Several auxiliary observations are collected in Section 5, and complete proofs of all results are presented in the concluding Section 6.

**3. Main results: acims for induced maps and extensions.** Given a nonsingular system  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$ , consider the family of acims defined as  $\mathfrak{M}(\mathfrak{S}) := \{\mu \ll \lambda : T\mu = \mu, T \text{ conservative w.r.t. } \mu\}$  together with the subfamily  $\mathfrak{M}_\sigma(\mathfrak{S}) := \{\mu \in \mathfrak{M}(\mathfrak{S}) : \mu \text{ } \sigma\text{-finite}\}$ . Also, let  $\tau$  be an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ , and  $\mathfrak{S}^*$  a  $\tau$ -trivialising extension with  $\tau$ -base  $Y^*$ . The following diagram informally depicts

the main results of this article, to be stated fully in the present section; each arrow indicates a well-defined map from one family of measures into another one.

$$\begin{array}{ccc}
\mathfrak{M}_\sigma(\mathfrak{S}^*|_{Y_\infty^*}) & \xleftarrow{\varphi_{Y^*}^* \times_{T^*} (\cdot)} & \mathfrak{M}_\sigma(\mathfrak{S}_{Y^*}^*) \\
\downarrow \pi(\cdot) & \xrightarrow{(\cdot)|_{Y^*}} & \downarrow \pi|_{Y^*}(\cdot) \quad \uparrow \pi|_{Y^*}^{-1}(\cdot) \\
\mathfrak{M}(\mathfrak{S}|_{Y_\infty}) & \xleftarrow{\tau \times_T (\cdot)} & \mathfrak{M}_\sigma(\mathfrak{S}^\tau)
\end{array}$$

In essence, Theorem 3.1 below asserts that this diagram does indeed commute. In the lower left corner, the subscript  $\sigma$  has been omitted for a reason: As it turns out, understanding the position of  $\mathfrak{M}_\sigma(\mathfrak{S}|_{Y_\infty})$  in this scheme is a main objective of the subsequent results. Assuming ergodicity throughout for convenience, Theorem 3.3 characterizes  $\mathfrak{M}_\sigma(\mathfrak{S}|_{Y_\infty}) \cap \tau \times_T (\mathfrak{M}_\sigma(\mathfrak{S}^\tau))$ . Via diagram chasing, this also yields a characterization of  $\mathfrak{M}_\sigma(\mathfrak{S}|_{Y_\infty}) \cap \pi \mathfrak{M}_\sigma(\mathfrak{S}^*|_{Y_\infty^*})$ , recorded in Corollary 3.5. Finally, Theorem 3.7 and Corollary 3.8 describe  $\mathfrak{M}_\sigma(\mathfrak{S}^\tau) \cap (\tau \times_T (\cdot))^{-1} \mathfrak{M}_\sigma(\mathfrak{S}|_{Y_\infty})$  and  $\mathfrak{M}_\sigma(\mathfrak{S}^*|_{Y_\infty^*}) \cap \pi^{-1} \mathfrak{M}_\sigma(\mathfrak{S}|_{Y_\infty})$ , respectively.

As indicated above, the key to exploiting the diagram is

**Theorem 3.1 (Invariant measures for induced maps and extensions).** *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular, and  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ . Also, let  $(X^*, \mathcal{A}^*, \lambda^*, T^*)$  be a  $\tau$ -trivialising extension with factor map  $\pi$  and  $\tau$ -base  $Y^*$ , and  $Y_\infty^* := \bigcap_{n \geq 0} \bigcup_{j \geq n} (T^*)^{-j} Y^*$ .*

- (i) *Assume that  $T^\tau$  preserves the  $\sigma$ -finite measure  $\nu \ll \lambda|_Y$  and is conservative w.r.t.  $\nu$ . Then  $T^*$  preserves a  $\sigma$ -finite measure  $\mu^* \ll \lambda^*|_{Y_\infty^*}$  that satisfies  $\pi|_{Y^*}(\mu^*|_{Y^*}) = \nu$  as well as  $\pi\mu^* = \tau \times_T \nu =: \mu$ . Moreover,  $T^*$  is conservative w.r.t.  $\mu^*$ , and  $\mu$  is  $T$ -invariant.*
- (ii) *Conversely, assume that  $T^*$  preserves the  $\sigma$ -finite measure  $\mu^* \ll \lambda^*|_{Y_\infty^*}$ , and that  $T^*$  is conservative w.r.t.  $\mu^*$ . Then  $T^\tau$  preserves the  $\sigma$ -finite measure  $\nu := \pi|_{Y^*}(\mu^*|_{Y^*}) \ll \lambda|_Y$ , and is conservative w.r.t.  $\nu$ . Moreover,  $\nu$  satisfies  $\tau \times_T \nu = \pi\mu^* =: \mu$ , and  $\mu$  is  $T$ -invariant.*

**Remark 3.2.** (i) In the situation of the theorem,  $T^*$  is conservative w.r.t.  $\mu^*$ . This immediately implies that  $T$  is conservative w.r.t.  $\mu$ .

(ii) In part (i) of the theorem, conservativity of  $T^\tau$  is only required to ensure conservativity of  $T^*$ . All other assertions are valid without this assumption.

(iii) Since the assumptions in Theorem 3.1(i) do not stipulate any particular further properties of  $T^*$ , the conclusion shows that in principle one  $\tau$ -trivialising extension is as good as any other as far as the lifting of  $\mu$  to such an extension is concerned. Nevertheless, some extensions may be easier to work with than others.

(iv) The measure  $\mu$  does not have to be  $\sigma$ -finite.

(v) It is clear that  $T^*$ -invariant measures supported outside the invariant set  $Y_\infty^*$  cannot, in general, be understood in terms of  $T_{Y^*}^*$  or, equivalently,  $T^\tau$ .

Under the assumption that  $\mu$  is a *finite* ergodic  $T$ -invariant measure, the main result of [18] provides a necessary and sufficient condition for  $\tau \times_T \nu = \mu$  to have a

solution  $\nu$ . The following theorem sharpens this criterion, and also generalizes it to  $\sigma$ -finite conservative situations. To formulate it, let  $(X, \mathcal{A}, \lambda, T)$  be a nonsingular system, and  $\tau$  an inducing time for  $T$  on  $Y$ . For  $Z \in \mathcal{A} \cap Y$  let  $\varphi_Z^\tau(x) := \inf\{n \geq 1 : (T^\tau)^n x \in Z\} \in \overline{\mathbb{N}}$ , the first return time of  $Z$  under  $T^\tau$ , and (suppressing the dependence on  $\tau$  in the notation)  $\theta_Z := \sum_{k=0}^{\varphi_Z^\tau - 1} \tau \circ (T^\tau)^k$ . If  $Z$  is a recurrent set for  $T^\tau$ , then  $\theta_Z$  is an inducing time for  $T$  on  $Z$  with  $(T^\tau)_Z = T^{\theta_Z}$  (compare [18, Thm.1.2]). Define

$$\vartheta_Z := \sum_{n=0}^{\theta_Z - 1} 1_Z \circ T^n \quad \text{on } Z. \quad (8)$$

Provided that  $Z$  is recurrent for  $T^\tau$ ,  $\vartheta_Z$  is an inducing time for  $T_Z$  on  $Z$  with  $\vartheta_Z \leq \theta_Z$  and  $(T^\tau)_Z = (T_Z)^{\vartheta_Z}$ .

**Theorem 3.3 (Solving  $\tau \times_T \nu = \mu$  with conservative ergodic acim  $\mu$ ).** *Let  $T$  be a conservative ergodic mp map on the  $\sigma$ -finite space  $(X, \mathcal{A}, \mu)$ , and  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ . Then the following statements are equivalent:*

- (i)  $\tau \times_T \nu = \mu$  has a  $\sigma$ -finite solution  $\nu \ll \mu|_Y$  for which  $T^\tau$  is mp and conservative w.r.t.  $\nu$ ;
- (ii) There exists a set  $Z \in \mathcal{A} \cap Y$  with the property that

$$0 < \int_Z \vartheta_Z d\mu < \infty,$$

where  $\vartheta_Z$  is given by (8).

**Remark 3.4.** (i) Observe that the relation  $(T^\tau)_Z = (T_Z)^{\vartheta_Z}$  uniquely determines  $\vartheta_Z(x)$  unless  $(T^\tau)_Z x$  is a periodic point of  $T_Z$  (and hence of  $T$ ). Consequently, if  $T$  is aperiodic, i.e.  $\lambda(\{x : T^n x = x \text{ for some } n\}) = 0$ , then  $\vartheta_Z$  is uniquely determined mod  $\lambda$  by  $(T^\tau)_Z = (T_Z)^{\vartheta_Z}$ .

(ii) In statement (ii) of the theorem, since  $\vartheta_Z \geq 1$ , positivity of  $\int_Z \vartheta_Z d\mu$  is equivalent to  $\mu(Z) > 0$ , whereas finiteness of the integral guarantees that  $Z$  is a recurrent set for  $T^\tau$  w.r.t.  $\mu$ .

Together, Theorems 3.1 and 3.3 identify part (ii) of Theorem 3.3 as a sharp integrability condition for a conservative ergodic  $\sigma$ -finite acim of  $\mathfrak{S}$  to lift to a particular  $\mathfrak{S}^*$ .

**Corollary 3.5 (Liftability via integrability).** *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular with a nonsingular extension  $(X^*, \mathcal{A}^*, \lambda^*, T^*)$ , and  $Y^* \in \mathcal{A}^*$  a recurrent set for  $T^*$  such that  $\pi|_{Y^*} : Y^* \rightarrow Y \in \mathcal{A}$  is invertible. Assume that  $\mu \ll \lambda|_{Y_\infty}$  is a  $\sigma$ -finite conservative ergodic acim for  $T$ . Then the following property is equivalent to both (i) and (ii) in Theorem 3.3 with  $\tau := \varphi_{Y^*}^* \circ \pi|_{Y^*}^{-1}$ :*

- (iii) There exists a  $\sigma$ -finite measure  $\mu^* \ll \lambda^*|_{Y_\infty^*}$  with  $\pi\mu^* = \mu$  such that  $T^*$  is mp and conservative w.r.t.  $\mu^*$ .

**Remark 3.6.** (i) Assume that  $\mu$  is finite. Then integrability of  $\tau$  w.r.t. the invariant measure, i.e.  $\int_Y \tau d\mu < \infty$ , is sufficient for liftability. Indeed, take  $Z = Y$  in statement (ii) of Theorem 3.3, then  $\vartheta_Z$  is integrable since  $\vartheta_Z \leq \tau$ . However, even for finite  $\mu$  this condition is not necessary, see [18, Ex.2.3].

(ii) For the case  $\mu(X) = \infty$  and  $\mu(Y) < \infty$ , Kac' formula shows that  $\int_Y \varphi_Y d\mu = \infty$ . A fortiori,  $\int_Y \tau d\mu = \infty$  for every inducing time  $\tau$  for  $T$  on  $Y$ . Nonetheless, the relativised integrability criterion of Theorem 3.3 in terms of  $\vartheta_Z$  remains meaningful in this situation. For example, when applied to  $\tau = \varphi_Y$  and  $Z = Y$ , one obtains

$\vartheta_Z = 1$ , which is integrable, so that the theorem simply asserts solvability of  $\varphi_Y \times_T \nu = \mu$ , a fact already contained in (2) and (3).

(iii) While  $\mu(X) = \infty$  and  $\mu(Y) < \infty$  together preclude  $\mu$ -integrability of  $\tau$ , severe restrictions on the degree of nonintegrability of  $\tau$ , as captured by the asymptotic behaviour of its *tail*  $\mu(Y \cap \{\tau > n\})$ , often have to be imposed in order for  $\mu$  to lift to a  $\tau$ -trivialising extension of a specific type. For example, [4, Cor.2.7] contains the following observation: Let  $\mathfrak{S} = (X, \mathcal{A}, \mu, T)$  be ergodic and mp with  $\mu(X) = \infty$ . Assume that  $\mathfrak{S}_i^* = (X_i^*, \mathcal{A}_i^*, \mu_i^*, T_i^*)$ ,  $i \in \{1, 2\}$ , are two mp Young towers over  $\mathfrak{S}$  with  $\pi_i \mu_i^* = \mu$ , so that, in particular, each  $\mathfrak{S}_i^*$  is a  $\tau_i$ -trivialising extension of  $\mathfrak{S}$  for a suitable inducing time  $\tau_i : Y_i \rightarrow \overline{N}$  on  $\mu(Y_i) < \infty$ . Then regular variation of  $\mu(Y_1 \cap \{\tau_1 > n\})$  with some index  $-\alpha$ , where  $\alpha \in (0, 1)$ , implies that  $\mu(Y_1 \cap \{\tau_1 > n\}) \sim \text{const} \cdot \mu(Y_2 \cap \{\tau_2 > n\})$  as  $n \rightarrow \infty$ . Thus  $\mu$  can only be lifted to a Young tower with a corresponding  $\tau$  that has a very specific tail behaviour. It may be worth noting that the latter observation also follows (via a Tauberian theorem) from [19, Thm.1] which asserts that *minimal wandering rates* (i.e. equivalence classes of the asymptotics of averaged tails, see [1] for authoritative definitions) are invariant under mp factor maps.

The remaining two results address the question of  $\sigma$ -finiteness of  $\mu$ , which is of importance in infinite-measure situations such as those considered in [1, 4, 17], recall also Remark 3.2.(iv). The first result offers an alternative representation of  $\tau \times_T \nu$  in terms of  $\nu$ , which in turn leads to an integrability criterion intimately related to that of Theorem 3.3.

**Theorem 3.7 (Alternative representation and  $\sigma$ -finiteness of  $\mu = \tau \times_T \nu$ ).** *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular,  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ , and assume that there is a  $\sigma$ -finite conservative ergodic invariant measure  $\nu \ll \lambda|_Y$  for  $T^\tau$ . Then for every  $Z \in \mathcal{A} \cap Y$  the conservative ergodic  $T$ -invariant measure  $\mu := \tau \times_T \nu \ll \lambda$  satisfies*

$$\mu(Z) = \int_Z \vartheta_Z d\nu,$$

where  $\vartheta_Z$  is given by (8). Hence,  $\mu$  is  $\sigma$ -finite iff there exists  $Z \in \mathcal{A} \cap Y$  with the property that

$$0 < \int_Z \vartheta_Z d\nu < \infty.$$

By virtue of Theorem 3.1, there is a corresponding criterion for an ergodic  $\sigma$ -finite acim  $\mu^*$  of an extension to project onto a  $\sigma$ -finite acim  $\mu$  for the factor.

**Corollary 3.8 (Projectability via integrability).** *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular with a nonsingular extension  $(X^*, \mathcal{A}^*, \lambda^*, T^*)$ , and  $Y^* \in \mathcal{A}^*$  a recurrent set for  $T^*$  with  $\pi|_{Y^*} : Y^* \rightarrow Y \in \mathcal{A}$  invertible. Assume that  $T^*$  preserves a  $\sigma$ -finite measure  $\mu^* \ll \lambda^*|_{Y_\infty^*}$ , and that  $T^*$  is conservative ergodic w.r.t.  $\mu^*$ . Let  $\tau := \varphi_{Y^*}^* \circ \pi|_{Y^*}^{-1}$ , and define  $\nu := \pi|_{Y^*}(\mu^*|_{Y^*})$ . Then the conservative ergodic invariant measure  $\mu := \pi\mu^* \ll \lambda$  of  $T$  is  $\sigma$ -finite iff there exists a set  $Z \in \mathcal{A} \cap Y$  with  $0 < \int_Z \vartheta_Z d\nu < \infty$ .*

**Remark 3.9.** (i) Viewing an induced map  $T^\tau$  through a  $\tau$ -trivialising extension as in this corollary, one finds that the formula for  $\mu(Z)$  in Theorem 3.7 is the classical Kac formula in disguise. Indeed, in the situation of the corollary one has

$$\vartheta_Z \circ \pi = \psi_{Z^*}^* \quad \text{a.e. on } Z^*,$$



where  $\psi_{Z^*}^*(x^*) := \inf\{n \geq 1 : (T_{\pi^{-1}Z}^*)^n x^* \in Z^*\}$  denotes the first return time of  $Z^* := Y^* \cap \pi^{-1}Z$  for the  $\mu^*|_{\pi^{-1}Z}$ -preserving conservative ergodic map  $T_{\pi^{-1}Z}^*$  on  $\pi^{-1}Z \subseteq X^*$ , so that  $\int_Z \vartheta_Z d\nu = \int_{Z^*} \psi_{Z^*}^* d\mu^* = \mu^*(\pi^{-1}Z) = \mu(Z)$  by Kac' formula.

(ii) Corollary 3.8 was motivated by [4, Thm.2.1] which starts from somewhat more restrictive assumptions. (Specifically, the extension is assumed to be a Young tower.) Under these assumptions,  $(Y, \mathcal{A} \cap Y, \lambda|_Y, T^\tau)$  is ergodic and has a finite acim  $\nu$  for which  $\log \frac{d\nu}{d\lambda}$  is bounded, hence  $\int_Z \vartheta_Z d\nu$  is finite iff  $\int_Z \vartheta_Z d\lambda$  is. Consequently, [4, Thm.2.1], which states that  $\mu$  is  $\sigma$ -finite iff  $\int_Z \vartheta_Z d\lambda < \infty$ , is a special case of Corollary 3.8.

**4. A classical example:  $\beta$ -transformations.** First steps towards the proofs of the main results will not be taken until Section 5 below. Strictly speaking, therefore, the present section is not essential for the development of this article. Its sole purpose is to illustrate the natural relation between general induced maps and extensions in the context of a truly classical example, namely the  $\beta$ -transformation. For the reader's convenience the discussion outlines, in this classical setup, the following typical scenario for an application of either construction: Given  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$ , try to find an induced system  $\mathfrak{S}^\tau$  or an extension  $\mathfrak{S}^*$  which is simpler than  $\mathfrak{S}$  in that it is known to possess an acim with certain desirable properties. Then use (5) or (6) to explicitly obtain a  $T$ -invariant measure which inherits some of these properties.

Recall that  $\mathfrak{S}$  is *piecewise invertible* if it comes with a countable measurable partition  $\zeta \pmod{\lambda}$  of  $X$  such that the restriction  $T|_Z : Z \rightarrow TZ$  to each *cylinder*  $Z \in \zeta$  is invertible. The partition is *(one-sided) Markov* if each  $TZ$  is measurable w.r.t.  $\zeta$ . In that case, if  $\nu \ll \lambda$  is either zero or equivalent to  $\lambda$  on each  $Z \in \zeta$ , then so is  $T\nu$ . This property enables consistent and effective coarse-graining through  $\zeta$ .

The classical examples herein are *piecewise affine interval maps* on  $X := [0, 1)$ , i.e. maps that are nonsingular w.r.t. Lebesgue measure  $\lambda$  and piecewise invertible with every cylinder  $Z$  a non-degenerate interval, and each  $T|_Z : Z \rightarrow TZ$  affine. If, for such a map  $T$ , every cylinder is *full*, meaning that  $TZ = X$  for every  $Z \in \zeta$  (so that, in particular, the partition is Markov), then  $T$  is easily seen to be  $\lambda$ -preserving and ergodic. However, within the family of  $\beta$ -transformations, those whose natural partition is Markov are exceptional, and one strives to regain the very convenient Markov property by constructing a suitable auxiliary system. Inducing provides one way of achieving this.

**Example 4.1 ( $\beta$ -transformations induced).** For every real  $\beta > 1$ , consider the  $\beta$ -transformation  $T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor$  on  $X = [0, 1)$ , with associated partition

$$\zeta := \left\{ \left[ 0, \frac{1}{\beta} \right), \left[ \frac{1}{\beta}, \frac{2}{\beta} \right), \dots, \left[ \frac{\lceil \beta \rceil - 2}{\beta}, \frac{\lceil \beta \rceil - 1}{\beta} \right), \left[ \frac{\lceil \beta \rceil - 1}{\beta}, 1 \right) \right\};$$

here  $\lfloor x \rfloor$  and  $\lceil x \rceil$  denote, respectively, the largest integer not larger and the smallest integer not smaller than  $x \in \mathbb{R}$ . For every  $n \geq 0$ , let  $\zeta_n$  be the family of maximal monotonicity intervals of  $T_\beta^n$ ; thus  $\zeta_0 = \{X\}$ ,  $\zeta_1 = \zeta$ , etc. Also, for every  $x \in X$  and  $n \in \mathbb{N}$ , denote by  $\zeta_n(x)$  the unique interval in  $\zeta_n$  containing  $x$ . With this, define  $\tau(x) := \inf\{n \geq 1 : \zeta_n(x) \text{ is full}\}$ . For an equivalent definition of  $\tau$ , let  $T_\beta^j 1 := \lim_{x \uparrow 1} T_\beta^j x \in (0, 1]$ ,  $j \geq 0$ . Then  $\tau(x) = n$  iff  $\sum_{j=1}^{n-1} \varepsilon_j \beta^{-j} \leq x < \sum_{j=1}^n \varepsilon_j \beta^{-j}$ , where  $\varepsilon_j := \lfloor \beta T_\beta^{j-1} 1 \rfloor$  and, as usual, the empty sum is interpreted as zero. From  $\sum_{j=1}^\infty \varepsilon_j \beta^{-j} \geq 1$ , it follows that  $\tau(x) < \infty$  for every  $x \in X$ , and  $\tau$  is an inducing

time for  $T_\beta$  on  $Y := X$ . Moreover,  $T_\beta^\tau$  is a piecewise affine map all of whose cylinders are full and hence is  $\lambda$ -preserving and ergodic. According to (5), the measure

$$\mu := \tau \times_{T_\beta} \lambda = \sum_{n \geq 0} T_\beta^n(\lambda|_{\{\tau > n\}}) \ll \lambda$$

is  $T_\beta$ -invariant (and easily seen to be finite), cf. also Lemma 5.6(i) below. Furthermore, due to (1),  $T_\beta$  is ergodic. For every  $n \geq 0$ , the iterate  $T_\beta^n$  maps  $\{\tau > n\} = [\sum_{j=1}^n \varepsilon_j \beta^{-j}, 1)$  affinely onto  $[0, T_\beta^n 1)$  with slope  $\beta^n$ , provided that  $\{\tau > n\}$  is not empty. Hence with  $N_\beta := \inf\{n \in \mathbb{N} : \{\tau > n\} = \emptyset\} \in \overline{\mathbb{N}}$ ,

$$\mu = \sum_{n=0}^{N_\beta-1} \beta^{-n} \lambda|_{[0, T_\beta^n 1)}, \quad (9)$$

which is the *Gelfond–Parry formula* for the  $T_\beta$ -invariant measure, see e.g. [5, 12]. Note that  $T_\beta \lambda = \lambda$  iff  $\beta$  is an integer, in which case  $\mu = \lambda$ , or equivalently iff  $N_\beta = 1$ . Also,  $N_\beta = \infty$  holds for all but countably many  $\beta > 1$ .

Alternatively, an acim may be obtained via a tower construction. In fact, various finer dynamical properties can be established this way, see e.g. [7, 8]. The following is an example of a canonical Markov extension in the sense of [10, 11]. It separates all different images of cylinder sets.

**Example 4.2 ( $\beta$ -transformations extended).** As in Example 4.1 let  $\zeta_n$  be the family of maximal monotonicity intervals of  $T_\beta^n$ , and for every  $n \geq 0$  consider the family of images  $\eta_n := \{T_\beta^n Z : Z \in \zeta_n\}$ . Clearly,  $\{X\} = \eta_0 \subset \eta_1 \subset \eta_2 \subset \dots$ . Since only the right-most cylinder in  $\zeta_n$  can have an image under  $T_\beta^n$  that is not already contained in  $\eta_j$  for some  $j < n$ , it follows that  $\eta_{n+1}$  contains at most one element more than  $\eta_n$ . Also, if  $\eta_{n+1} = \eta_n$  for some  $n$  then  $\eta_j = \eta_n$  for all  $j \geq n$ . Consequently, define  $N_\beta^* := \inf\{n \in \mathbb{N} : \eta_n = \eta_{n-1}\} \in \overline{\mathbb{N}}$ , let  $X_0 := X$ , and for every  $0 < n < N_\beta^*$  denote by  $X_n$  the unique interval in  $\eta_n \setminus \eta_{n-1}$ . Using the notation introduced in Example 4.1, it is not hard to see that  $N_\beta^* = N_\beta$ , and  $X_n = [0, T_\beta^n 1)$  for all  $0 \leq n < N_\beta$ . With this, let  $X^* := \bigcup_{n=0}^{N_\beta-1} (X_n \times \{n\}) \subset [0, 1) \times \mathbb{N}_0$ , equipped with the obvious version  $\lambda^*$  of Lebesgue measure, i.e.  $\lambda^* := (\lambda \times \#)|_{X^*}$  where  $\#$  is the counting measure, and  $\pi : X^* \rightarrow X$  the projection onto the first factor, that is,  $\pi(x, n) := x$  for all  $(x, n) \in X^*$ . The family  $\zeta^* := \{(Z \cap X_n) \times \{n\} : Z \in \zeta, 0 \leq n < N_\beta\}$  forms a partition of  $X^*$ . Moreover, define  $T^* : X^* \rightarrow X^*$  according to

$$T^*(x, n) := \begin{cases} (T_\beta x, 0) & \text{if } \zeta(x) \text{ is full,} \\ (T_\beta x, n+1) & \text{otherwise.} \end{cases}$$

Clearly,  $T^*$  is a nonsingular extension of  $T_\beta$  and maps each element of  $\zeta^*$  affinely onto a set of the form  $X_n \times \{n\}$ . Thus  $(X^*, T^*)$  is a Markov extension, and the family of measures of the form  $\sum_{n=0}^{N_\beta-1} c_n \lambda^*|_{X_n \times \{n\}}$  is closed under  $T^*$ . It is readily confirmed that every  $T^*$ -invariant measure in this family is proportional to  $\mu^* := \sum_{n=0}^{N_\beta-1} \beta^{-n} \lambda^*|_{X_n \times \{n\}}$ . Projection onto  $X$  gives the  $T_\beta$ -invariant measure  $\pi \mu^* = \sum_{n=0}^{N_\beta-1} \beta^{-n} \lambda|_{X_n} \ll \lambda$  which again is the Gelfond–Parry formula (9).

By ergodicity, the finite invariant  $\lambda$ -a.c. measure for  $T_\beta$  is unique up to a multiplicative constant. Therefore any method identifying an acim must lead to the same result. However, the two natural constructions above are intimately related.

**Example 4.3 ( $\beta$ -transformations, continued).** The base  $Y^* := X \times \{0\}$  of the extension of Example 4.2 is an embedded copy of the recurrent set  $Y = X$  in

Example 4.1, as  $\pi|_{Y^*} : Y^* \rightarrow Y$  trivially is invertible as a nonsingular map. The iterates  $(T^*)^n$  map each point  $x^* \in Y^*$  into copies of  $T^n \zeta_n(\pi x^*)$  which are kept disjoint from  $Y^*$  until the first time they coincide with  $Y^*$ , that is, until  $\zeta_n(\pi x^*)$  is a full cylinder. Hence  $(T^*)^n x^*$  first returns to  $Y^*$  at time  $\tau(\pi x^*)$  or, in other words,  $\varphi_{Y^*}(x^*) = \tau \circ \pi(x^*)$  holds for every  $x^* \in Y^*$ . That is, the extension of Example 4.2 trivialises the induced map of Example 4.1.

Finally, two simple variations of the above examples are considered in

**Example 4.4 ( $\beta$ -transformations, once again).** As in Examples 4.1 and 4.2 let  $(X, \mathcal{A}, \lambda)$  be the interval  $[0, 1)$  equipped with its Borel  $\sigma$ -algebra and Lebesgue measure, and consider the  $\beta$ -transformation  $T_\beta : X \rightarrow X$ .

(i) Pick  $\beta = \frac{1+\sqrt{5}}{2}$ . For this particular value of  $\beta$ , where  $T_\beta 1 = \beta - 1$  coincides with the discontinuity point  $\beta^{-1}$ , consider  $\tilde{\tau} : X \rightarrow \mathbb{N}$  with

$$\tilde{\tau}(x) = \begin{cases} 1 & \text{if } 0 \leq x < \beta^{-1}, \\ 3 & \text{otherwise.} \end{cases}$$

Trivially,  $\tilde{\tau}$  is an inducing time for  $T_\beta$  on  $Y = X$ . However, the nonsingular extension provided by Example 4.2 is *not*  $\tilde{\tau}$ -trivialising in this case, as  $\tilde{\tau} \circ \pi(x, 0) = 3 \neq 2 = \varphi_{X \times \{0\}}^*(x, 0)$  whenever  $\beta^{-1} \leq x < 1$ . On the other hand, choosing  $\tilde{X} := X \times \{0, 2\} \cup [0, \beta^{-1}) \times \{1\}$  and

$$\tilde{T}(x, n) = \begin{cases} (T_\beta(x), 0) & \text{if } n = 0 \text{ and } 0 \leq x < \beta^{-1}, \\ (T_\beta(x), n + 1 - 3\lfloor \frac{1}{3}(n + 1) \rfloor) & \text{otherwise,} \end{cases}$$

yields a  $\tilde{\tau}$ -trivialising extension of  $(X, \mathcal{A}, \lambda, T_\beta)$  that is also Markov.

(ii) Let  $\beta = 2$  and  $(X^*, \mathcal{A}^*, \lambda^*) = \otimes_{j=1}^2 (X, \mathcal{A}, \lambda)$ . With the so-called *baker's map*  $T^* : X^* \rightarrow X^*$ , defined as

$$T^*(x, y) = \begin{cases} (T_2(x), \frac{1}{2}y) & \text{if } 0 \leq x < \beta^{-1}, \\ (T_2(x), \frac{1}{2}(1 + y)) & \text{otherwise,} \end{cases}$$

$(X^*, \mathcal{A}^*, \lambda^*, T^*)$  is a version of the *natural extension* of  $(X, \mathcal{A}, \lambda, T_2)$ , see e.g. [13], and clearly constitutes a nonsingular extension as well. By Fubini's theorem, however,  $X^*$  does not contain any embedded set  $Y^*$  of positive measure. In particular, this extension is *not*  $\tau$ -trivialising for any inducing time  $\tau$  for  $T_2$ .

**5. Lemmas about induced systems, and a functorial property of  $\times_T$ .** In preparation for the proofs of the main results in Section 6 below, this section collects several basic facts for which no pertinent reference is known to the authors. These facts may be of independent interest beyond their usage here.

**More on first return maps.** Let  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$  be a nonsingular system,  $Y \in \mathcal{A}^+$  some recurrent set for  $T$ , and  $\nu \ll \lambda|_Y$ . For the first return time  $\tau = \varphi_Y$ , the measure  $\tau \times_T \nu = \varphi_Y \times_T \nu$  coincides with  $\mu$  as defined in (2). Moreover,

$$(\varphi_Y \times_T \nu)|_Y = \nu \quad \text{on } Y, \tag{10}$$

since  $Y \cap \{\varphi_Y > n\} \cap T^{-n}A = \emptyset$  for all  $n \geq 1$  whenever  $A \in \mathcal{A} \cap Y$ . It is well known that if  $\mu(Y) < \infty$  for an invariant measure  $\mu$ , then  $\mu|_Y$  is  $T_Y$ -invariant, and  $\mu$  is determined (on  $Y_\infty$ ) by  $\mu|_Y$  via the above construction, see (2) and (3). The following is a more general version of this principle.

**Lemma 5.1 (Invariant measures and their restrictions).** *Let  $(X, \mathcal{A}, \mu, T)$  be measure preserving, and assume that  $Y \in \mathcal{A}^+$  is recurrent. Then*

$$\mu \geq \varphi_Y \times_T (\mu|_Y) \quad \text{on } X, \quad (11)$$

and

$$\mu = \varphi_Y \times_T (\mu|_Y) \quad \text{on } Y_\infty \iff \mu|_Y \text{ is } T_Y\text{-invariant.} \quad (12)$$

If  $\mu|_Y$  is  $\sigma$ -finite and  $T$  is conservative on  $Y_\infty$ , then  $\mu|_Y$  is  $T_Y$ -invariant, and hence  $\mu|_{Y_\infty}$  is uniquely determined by  $\mu|_Y$ .

*Proof.* An inductive argument based on the decomposition  $T^{-1}(Y^c \cap \{\varphi_Y > n\}) = (Y \cap \{\varphi_Y > n+1\}) \cup (Y^c \cap \{\varphi_Y > n+1\})$ ,  $n \geq 0$ , shows that, for all  $A \in \mathcal{A}$  and  $N \geq 0$ ,

$$\mu(A) = \sum_{n=0}^N \mu(Y \cap \{\varphi_Y > n\} \cap T^{-n}A) + \mu(Y^c \cap \{\varphi_Y > N\} \cap T^{-N}A).$$

Letting  $N \rightarrow \infty$  gives  $\mu \geq \varphi_Y \times_T (\mu|_Y)$ .

To verify (12), assume first that  $\mu = \varphi_Y \times_T (\mu|_Y)$  on  $Y_\infty$ , and take  $A \in \mathcal{A} \cap Y$ . Then

$$\begin{aligned} \mu(A) &= \mu(T^{-1}A) = (\varphi_Y \times_T (\mu|_Y))(T^{-1}A) \\ &= \sum_{n \geq 0} \mu\left(Y \cap \{\varphi_Y > n\} \cap T^{-(n+1)}A\right) \\ &= \sum_{n \geq 0} \mu\left(Y \cap \{\varphi_Y = n+1\} \cap T^{-(n+1)}A\right) \\ &= \mu(T_Y^{-1}A). \end{aligned}$$

For the reverse implication, assume that  $\mu|_Y$  is  $T_Y$ -invariant. Then (2) shows that  $\tilde{\mu} := \varphi_Y \times_T (\mu|_Y) \ll \lambda|_{Y_\infty}$  is  $T$ -invariant with  $\tilde{\mu}|_Y = \mu|_Y$ . In view of (11), the measure  $\eta := \mu - \tilde{\mu} \ll \lambda|_{Y_\infty}$  is  $T$ -invariant and vanishes on  $Y$ . Due to the definition of  $Y_\infty$ , this implies  $\eta = 0$ , as claimed.

The statement about conservative maps is contained in [6, Satz 8].  $\square$

**Remark 5.2.** (i) If  $T$  is not conservative on  $Y_\infty$ , then  $\mu|_Y$  may not be  $T_Y$ -invariant, as is illustrated by the map  $T : x \mapsto x+1$  on  $X := \mathbb{Z}$  with the  $\sigma$ -finite  $\mu := \#$ , and  $Y := \mathbb{N}$  (a recurrent set). For further information regarding acims of systems which are not conservative see e.g. [2].

(ii) Conservativity is not always necessary for  $\mu|_Y$  to be  $T_Y$ -invariant: To see this, let  $X := \mathbb{Z} \times \{-1, 1\}$ ,  $\mu := \#$ , and  $T(x, y) := (x+1, -y)$  which is clearly not conservative. Nonetheless, the set  $Y := \mathbb{Z} \times \{1\}$  is recurrent, and  $\mu|_Y$  is  $T_Y$ -invariant. In fact, the proof of the lemma shows that  $\mu|_Y$  is  $T_Y$ -invariant whenever  $\varphi_Y$  is bounded.

The following result shows that in  $\sigma$ -finite measure preserving situations, conservativity can be checked using first return maps. Maharam's classical recurrence theorem [1, Thm.1.1.7] states that if  $\mathfrak{S} = (X, \mathcal{A}, \mu, T)$  is mp and  $Y \in \mathcal{A}^+$  is a recurrent set with  $\mu(Y) < \infty$ , then  $\mathfrak{S}|_{Y_\infty}$  is conservative. Since, under these assumptions,  $\mathfrak{S}_Y$  is automatically conservative (being a finite-measure preserving system), the next lemma is a generalization of the classical result.

**Lemma 5.3 (Conservativity via first return maps).** *Let  $\mathfrak{S} = (X, \mathcal{A}, \mu, T)$  be measure preserving, and assume that  $\mu$  is  $\sigma$ -finite on the recurrent set  $Y \in \mathcal{A}^+$ . Then  $\mathfrak{S}_Y$  is conservative iff  $\mathfrak{S}|_{Y_\infty}$  is conservative.*

*Proof.* Assume that  $\mathfrak{S}_Y$  is conservative. To prove that  $T$  is conservative on  $Y_\infty$  it suffices to show that  $Y_\infty = \bigcup_{j \geq 1} Z_{j,\infty} \bmod \mu$ , where, for every  $j$ ,  $Z_{j,\infty} \in \mathcal{A}$  is  $T$ -invariant and  $T$  is conservative on  $Z_{j,\infty}$ . Now  $Y = \bigcup_{j \geq 1} Z_j$  for suitable  $Z_j \in \mathcal{A}$  satisfying  $0 < \mu(Z_j) < \infty$ . As  $T_Y$  is conservative, each  $Z_j$  is a recurrent set for  $T_Y$ , and hence also for  $T$ . Therefore, each  $Z_{j,\infty} := \bigcup_{n \geq 0} T^{-n} Z_j$  is  $T$ -invariant, and it is clear that  $Y_\infty = \bigcup_{n \geq 0} T^{-n} Y = \bigcup_{j \geq 1} Z_{j,\infty}$  holds  $\bmod \mu$ . Observe then that  $Z_j$  is a sweep-out set of finite measure for the  $\mu|_{Z_{j,\infty}}$ -preserving map  $T|_{Z_{j,\infty}}$ . By Maharam's recurrence theorem,  $T$  is indeed conservative on each  $Z_{j,\infty}$ , as required. The reverse implication is standard, see e.g. [1, Prop.1.5.1].  $\square$

**Remark 5.4.** If  $T$  is only assumed to be nonsingular, rather than mp, then conservativity of  $T_Y$  no longer implies conservativity of  $T$ , as is illustrated by  $T : x \mapsto \max(1, x - 1)$  on  $X := \mathbb{N}$  with  $\sigma$ -finite  $\mu := \#$ , and  $Y := \{1\}$ . A similarly simple example shows that  $\sigma$ -finiteness of  $\mu$  is essential here too. Indeed, take  $X, Y$  and  $T$  as before, and let  $\mu(A) := \infty$  unless  $A = \emptyset$ . Then  $T$  preserves  $\mu$ ,  $Y$  is recurrent, and  $T_Y$  is conservative, whereas clearly  $T$  is not conservative on  $Y_\infty = X$ .

**More on general induced transformations.** Let  $\mathfrak{S} = (X, \mathcal{A}, \lambda, T)$  be a nonsingular system, and  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ . Note that  $\tau$  can always be represented in terms of the *successive return times*  $\varphi_{Y,1} < \varphi_{Y,2} < \varphi_{Y,3} < \dots$  of  $Y$ , given by  $\varphi_{Y,n} := \sum_{j=0}^{n-1} \varphi_Y \circ T_Y^j$ ,  $n \geq 1$ . In fact, there exists a measurable function  $\rho : Y \rightarrow \bar{\mathbb{N}}$ , finite a.e., such that  $\tau(x) = \varphi_{Y,\rho(x)}(x)$  for a.e.  $x \in Y$ , and hence  $T^\tau = (T_Y)^\rho$ , see [18, Rem.4.2].

By straightforward calculation one obtains a *generalised Kac formula* for the total mass of  $\tau \times_T \nu$  since, for every measure  $\nu \ll \lambda|_Y$  on  $(Y, \mathcal{A} \cap Y)$ ,

$$\tau \times_T \nu(X) = \sum_{n \geq 0} \nu(Y \cap \{\tau > n\}) = \int_Y \tau \, d\nu. \quad (13)$$

A useful counterpart to (13) is contained in

**Lemma 5.5 (Weight of  $Y$  under  $\tau \times_T \nu$ ).** *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular, and  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$  with  $T^\tau = (T_Y)^\rho$ . If  $\nu$  is a measure on  $(Y, \mathcal{A} \cap Y)$  with  $\nu \ll \lambda|_Y$ , then*

$$\tau \times_T \nu(Y) = \int_Y \rho \, d\nu. \quad (14)$$

*Proof.* Start from the definition of  $\tau \times_T \nu$ , decompose  $Y$  according to the value of  $\rho$ , and then according to the values of the successive return-times  $\varphi_{Y,1} < \varphi_{Y,2} < \dots < \varphi_{Y,\rho}$ . Observe that, given  $r \in \mathbb{N}$  and natural numbers  $k_1 < k_2 < \dots < k_r$ , the set

$$Y \cap \{\rho = r\} \cap \{\varphi_{Y,1} = k_1\} \cap \dots \cap \{\varphi_{Y,r} = k_r\} \cap T^{-n} Y$$

can, for  $n < k_r$ , only be non-empty if  $n \in \{0, k_1, \dots, k_{r-1}\}$ , in which case it equals  $Y \cap \{\rho = r\} \cap \{\varphi_{Y,1} = k_1\} \cap \dots \cap \{\varphi_{Y,r} = k_r\}$  and is contained in  $\{\tau = k_r\}$ .

Consequently,

$$\begin{aligned}
\tau \times_T \nu(Y) &= \sum_{n \geq 0} \nu(Y \cap \{\tau > n\} \cap T^{-n}Y) \\
&= \sum_{r \geq 1} \sum_{k_1 < \dots < k_r} \sum_{n=0}^{k_r-1} \nu\left(Y \cap \{\rho = r\} \cap \bigcap_{j=1}^r \{\varphi_{Y,j} = k_j\} \cap T^{-n}Y\right) \\
&= \sum_{r \geq 1} \sum_{k_1 < \dots < k_r} r \cdot \nu\left(Y \cap \{\rho = r\} \cap \bigcap_{j=1}^r \{\varphi_{Y,j} = k_j\}\right) \\
&= \sum_{r \geq 1} r \cdot \nu(Y \cap \{\rho = r\}),
\end{aligned}$$

which proves (14).  $\square$

As far as absolutely continuous invariant measures are concerned, it is most important that (2) and (3) extend, in a natural way, to general inducing times.

**Lemma 5.6** ( *$T^\tau$ -invariance of  $\nu$  vs.  $T$ -invariance of  $\tau \times_T \nu$* ). *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular,  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ , and  $\nu \ll \lambda|_Y$  any measure on  $(Y, \mathcal{A} \cap Y)$ . Then the following implications hold with  $\mu := \tau \times_T \nu \ll \lambda|_{Y_\infty}$ :*

- (i) *If  $\nu$  is  $T^\tau$ -invariant, then  $\mu$  is  $T$ -invariant;*
- (ii) *If  $\mu$  is  $T$ -invariant and  $\sigma$ -finite, then  $\nu$  is  $T^\tau$ -invariant and  $\sigma$ -finite.*

*Proof.* Assertion (i) is well-known, see e.g. [14, Eqn.(1.3)]; it follows immediately from the fact that, for every  $A \in \mathcal{A}$ ,

$$(\tau \times_T \nu)(T^{-1}A) = \nu((T^\tau)^{-1}(Y \cap A)) + \sum_{n \geq 1} \nu(Y \cap \{\tau > n\} \cap T^{-n}A). \quad (15)$$

To prove (ii), denote by  $\eta(A)$  the right-most sum in (15), and assume that  $\tau \times_T \nu$  is  $T$ -invariant. In this case, for every  $A \in \mathcal{A}$ ,

$$(\tau \times_T \nu)(A) = \nu((T^\tau)^{-1}(Y \cap A)) + \eta(A) = \nu(Y \cap A) + \eta(A),$$

by the very definition of  $\tau \times_T \nu$ . Since  $\eta \leq (\tau \times_T \nu)$ , the equality

$$\nu((T^\tau)^{-1}A) = \nu(A) \quad (16)$$

holds for all  $A \in \mathcal{A} \cap Y$  with  $(\tau \times_T \nu)(A) < \infty$ . In case  $\tau \times_T \nu$  is merely  $\sigma$ -finite on  $A \in \mathcal{A} \cap Y$ , write  $A = \bigcup_{j \geq 1} A_j$ , where the  $A_j$  are disjoint and  $(\tau \times_T \nu)(A_j) < \infty$  for every  $j$ . Since, for each  $j$ , (16) holds with  $A$  replaced by  $A_j$ , it also holds for  $A$  itself. Thus  $\nu$  is  $T^\tau$ -invariant, and  $\sigma$ -finiteness of  $\nu$  is clear as  $\nu \leq \mu$ .  $\square$

**Remark 5.7.** Without  $\sigma$ -finiteness of  $\tau \times_T \nu$ , Lemma 5.6(ii) is false in general. To see this, consider for instance the map  $T : x \mapsto 2 \min(x, 1-x)$  on  $X := [0, 1]$  with Lebesgue measure  $\lambda$ , and choose any  $\nu \ll \lambda$  such that  $\log \frac{d\nu}{d\lambda}$  is bounded. For every  $n \in \mathbb{N}$ , let  $I_n := (2^{-n}, 2^{1-n}]$ , and define  $\tau(x) := 2^n$  for all  $x \in I_n$ . Trivially,  $\tau$  is an inducing time for  $T$  on  $Y := X$ , and it is readily confirmed that  $\tau \times_T \nu(A)$  equals  $\infty$  or 0 if, respectively,  $\lambda(A) > 0$  or  $\lambda(A) = 0$ . Thus  $\tau \times_T \nu$  is  $T$ -invariant but, as  $T^\tau$  is ergodic w.r.t.  $\lambda$ , the measure  $\nu$  is  $T^\tau$ -invariant only if  $\frac{d\nu}{d\lambda}$  is constant.<sup>2</sup>

**Measure construction and towers.** A natural functorial property of the measure extending operation  $\times_T$  provides the final crucial link between the two types of auxiliary constructions studied in this article.

<sup>2</sup>**Two corrections to [18]:** The folklore fact (3) was misrepresented in [18] in that the condition  $\mu(Y) < \infty$  was left out. Similarly, Lemma 5.6(ii) was quoted incorrectly in [18, Prop.1.1], with the assumption of  $\sigma$ -finiteness missing.

**Lemma 5.8 (Factor maps and  $\times_T$ ).** *Let  $(X, \mathcal{A}, \lambda, T)$  be nonsingular, and  $\tau$  an inducing time for  $T$  on  $Y \in \mathcal{A}^+$ . Assume that  $(X^*, \mathcal{A}^*, \lambda^*, T^*)$  is a  $\tau$ -trivialising extension with factor map  $\pi$  and  $\tau$ -base  $Y^*$ . Then, for every measure  $\eta \ll \lambda|_Y$  on  $\mathcal{A} \cap Y$ ,*

$$\tau \times_T \eta = \pi((\tau \circ \pi) \times_{T^*} (\pi|_{Y^*}^{-1} \eta)). \quad (17)$$

*Proof.* By (7),  $\tau \circ \pi$  is an inducing time for  $T^*$  on  $Y^*$ , which implies that the right-hand expression in (17) does make sense. Pick any  $A \in \mathcal{A}$ , and use the definition of  $\times_{T^*}$  as well as the factor property of  $\pi$  to see that indeed

$$\begin{aligned} \pi((\tau \circ \pi) \times_{T^*} (\pi|_{Y^*}^{-1} \eta))(A) &= \sum_{n \geq 0} \pi|_{Y^*}^{-1} \eta(Y^* \cap \{\tau \circ \pi > n\} \cap (T^*)^{-n} \pi^{-1} A) \\ &= \sum_{n \geq 0} \pi|_{Y^*}^{-1} \eta(Y^* \cap \pi^{-1}(\{\tau > n\} \cap T^{-n} A)) \\ &= \sum_{n \geq 0} \eta(Y \cap \{\tau > n\} \cap T^{-n} A) \\ &= \tau \times_T \eta(A). \end{aligned}$$

□

**6. Proofs of the main results.** With the classical facts recalled in Section 2 and the specific lemmas provided in Section 5, all the required ingredients have been assembled for a

*Proof of Theorem 3.1.* To prove (i), assume that  $T^\tau$  preserves the  $\sigma$ -finite measure  $\nu \ll \lambda|_Y$ . As  $\pi|_{Y^*}$  is invertible, (7) implies that the first return map  $T_{Y^*}^*$  of  $T^*$  on  $Y^*$  preserves the  $\sigma$ -finite measure  $\pi|_{Y^*}^{-1} \nu \ll \lambda^*|_{Y^*}$ , since

$$T_{Y^*}^*(\pi|_{Y^*}^{-1} \nu) = \pi|_{Y^*}^{-1}(T^\tau \nu) = \pi|_{Y^*}^{-1} \nu \quad \text{on } Y^*.$$

According to (2), the  $T_{Y^*}^*$ -invariant measure  $\pi|_{Y^*}^{-1} \nu$  is the restriction to  $\mathcal{A}^* \cap Y^*$  of the  $T^*$ -invariant measure

$$\mu^* := \varphi_{Y^*}^* \times_{T^*} (\pi|_{Y^*}^{-1} \nu) = (\tau \circ \pi) \times_{T^*} (\pi|_{Y^*}^{-1} \nu) \ll \lambda^*|_{Y_\infty^*}.$$

But then  $\pi\mu^* = \tau \times_T \nu$  is immediate from Lemma 5.8, applied to  $\eta = \nu$ , and  $\pi|_{Y^*} \mu^* = \nu$  is clear since  $\mu^*|_{Y^*} = \pi|_{Y^*}^{-1} \nu$ . Also, by (4), the measure  $\mu^*$  is  $\sigma$ -finite. Finally,  $T^\tau$  is conservative w.r.t.  $\nu$  iff  $T_{Y^*}^*$  is conservative w.r.t.  $\pi|_{Y^*}^{-1} \nu$ , and due to Lemma 5.3 the latter implies that  $T^*$  is conservative w.r.t.  $\mu^*$ .

To show (ii), assume that  $T^*$  preserves the conservative  $\sigma$ -finite measure  $\mu^* \ll \lambda^*|_{Y_\infty^*}$ . According to Lemma 5.1 this implies that  $T_{Y^*}^*$  preserves  $\mu^*|_{Y^*}$ , and that  $\mu^* = \varphi_{Y^*}^* \times_{T^*} (\mu^*|_{Y^*})$ . Define  $\nu := \pi|_{Y^*} (\mu^*|_{Y^*})$ , the  $\pi$ -image of  $\mu^*|_{Y^*}$  on  $\mathcal{A} \cap Y$ , and observe that (7) implies that  $\nu$  is  $T^\tau$ -invariant, as

$$T^\tau \nu = T^\tau (\pi|_{Y^*} \mu^*) = \pi|_{Y^*} (T_{Y^*}^* \mu^*|_{Y^*}) = \pi|_{Y^*} \mu^*|_{Y^*} = \nu \quad \text{on } Y.$$

By Lemma 5.8,  $\mu^*$  satisfies  $\pi\nu^* = \tau \times_T \nu$ , with  $\mu := \pi\mu^* \ll \lambda$  obviously being  $T$ -invariant. □

Combining several observations from [18] quite directly leads to a

*Proof of Theorem 3.3.* Assume first that (i) holds. By  $\sigma$ -finiteness of the two measures  $\mu, \nu$ , there exists  $Y' \in \mathcal{A} \cap Y$  such that  $\mu(Y'), \nu(Y') \in (0, \infty)$ , and  $\mu, \nu$  are equivalent on  $Y'$ . By conservativity of  $T^\tau$  w.r.t.  $\nu$ , the first return map  $(T^\tau)_{Y'}$  is defined a.e. on  $Y'$ . Therefore there is a (unique) inducing time  $\vartheta_{Y'}$  for  $T_{Y'}$  on  $Y'$  for which  $(T^\tau)_{Y'} = (T_{Y'})^{\vartheta_{Y'}}$  a.e. on  $Y'$ . Now [18, Prop.4.1] applies to show that

$\vartheta_{Y'} \times_{T_{Y'}} \nu' = \mu|_{Y'}$  has a solution  $\nu'$ . On the other hand, (3) and (1) ensure that  $T_{Y'}$  is ergodic and mp for the finite measure  $\mu|_{Y'}$ . Consequently, [18, Thm.1.2] applies to  $T_{Y'}$  and the inducing time  $\vartheta_{Y'}$ , proving that it is possible to choose  $Z \in \mathcal{A} \cap Y'$  with the property that  $0 < \mu(Z) < \infty$  and  $\int_Z \theta_Z d\mu < \infty$ , where  $\theta_Z$  is such that  $(T_{Y'})^{\theta_Z} = ((T_{Y'})^{\vartheta_{Y'}})_Z$  a.e. on  $Z$ . Now recall the definition of  $\vartheta_{Y'}$  to see that  $(T_{Y'})^{\theta_Z} = ((T^\tau)_{Y'})_Z = (T^\tau)_Z$ . Therefore,  $(T_{Y'})^{\theta_Z} = (T_Z)^{\vartheta_Z}$  a.e. on  $Z$ , and since  $Z \subseteq Y'$ , the latter implies that  $\vartheta_Z \leq \theta_Z$  a.e. Hence,  $\int_Z \vartheta_Z d\mu \leq \int_Z \theta_Z d\mu < \infty$ , establishing (ii).

Assume in turn that (ii) is satisfied. As  $\mu$  is conservative,  $Z$  is a recurrent set for  $T$  w.r.t.  $\mu$ . Assumption (ii) means, in particular, that  $\vartheta_Z < \infty$   $\mu$ -a.e. on  $Z$ , whence  $Z$  is a recurrent set for the nonsingular map  $T^\tau$  on  $(Y, \mathcal{A} \cap Y, \mu|_Y)$ . By (3) and (1),  $T_Z$  is ergodic and mp on the finite measure space  $(Z, \mathcal{A} \cap Z, \mu|_Z)$ . Now  $\vartheta_Z$  is an inducing time for  $T_Z$  on  $Z$  with  $\int_Z \theta_Z d\mu < \infty$ . Hence, in view of [18, Thm.1.1],  $\vartheta_Z \times_{T_Z} \tilde{\nu} = \mu|_Z$  has a solution  $\tilde{\nu}$ , which is necessarily a finite absolutely continuous measure since  $\tilde{\nu} \leq \mu|_Z$ . Moreover,  $\tilde{\nu}$  is invariant under  $(T_Z)^{\vartheta_Z} = (T^\tau)_Z$ , see part (ii) of Lemma 5.6. Let  $\varphi_Z^\tau(x) := \inf\{j \geq 1 : (T^\tau)^j x \in Z\}$  and observe that [18, Prop.4.1] applies to show that  $\nu := \varphi_Z^\tau \times_{T^\tau} \tilde{\nu}$  solves  $\mu = \tau \times_T \nu$ . Use part (ii) of Lemma 5.6 again to see that  $\nu$  is indeed a  $\sigma$ -finite acim for  $T^\tau$  on  $Y$ . By construction of  $\nu$ ,  $Y \subseteq Z_\infty^\tau := \bigcup_{n \geq 1} (T^\tau)^{-n} Z \bmod \nu$ , and since  $\nu(Z) = \tilde{\nu}(Z) < \infty$ , Maharam's recurrence theorem [1, Thm.1.1.7] now shows that  $T^\tau$  is conservative w.r.t. its invariant measure  $\nu$ . This proves (i).  $\square$

As indicated earlier, a first corollary now follows immediately.

*Proof of Corollary 3.5.* Simply note that the nonsingular extension  $(X^*, \mathcal{A}^*, \lambda^*, T^*)$  is  $\tau$ -trivialising with base  $Y^*$ . Theorem 3.1 thus shows that (iii) is equivalent to statement (i) of Theorem 3.3. Equivalence to Theorem 3.3(ii) is immediate, as all assumptions of Theorem 3.3 are met.  $\square$

The question of  $\sigma$ -finiteness of  $\mu = \tau \times_T \nu$  is clarified by the

*Proof of Theorem 3.7.* In view of the preceding arguments, it only remains to verify the formula for  $\mu$ . Note first that for  $Z = Y$ , Lemma 5.5 gives  $\mu(Y) = \int_Y \vartheta_Y d\nu$ . In general, given  $Z \in \mathcal{A} \cap Y$  with  $\mu(Z) > 0$ , let  $\sigma, \vartheta_Z : X \rightarrow \mathbb{N}$  be such that  $T^\sigma = (T^\tau)_Z = (T_Z)^{\vartheta_Z}$  holds  $\lambda$ -a.e. on  $Z$ . Applying Lemma 5.5 to  $T$  with inducing time  $\sigma$  and set  $Z$  yields

$$\sigma \times_T (\nu|_Z)(Z) = \int_Z \vartheta_Z d\nu.$$

From the  $T^\tau$ -invariance of  $\nu$ , it follows that  $\varphi_Z^\tau \times_{T^\tau} \nu|_Z = \nu$ , where  $\varphi_Z^\tau(x) := \inf\{n \geq 1 : (T^\tau)^n x \in Z\}$  whenever  $x \in Z$ , and  $\varphi_Z^\tau(x) := 0$  otherwise; in particular, therefore,  $T^\sigma = (T^\tau)^{\varphi_Z^\tau}$ . Now apply the chain rule of [18, Lem.4.1] to obtain

$$\sigma \times_T \nu|_Z(Z) = \tau \times_T (\varphi_Z^\tau \times_{T^\tau} \nu|_Z)(Z) = \tau \times_T \nu(Z) = \mu(Z).$$

Finally, being conservative ergodic,  $\mu$  is  $\sigma$ -finite iff  $Y$  contains some measurable  $Z$  with  $0 < \mu(Z) < \infty$ .  $\square$

The article concludes by providing a

*Proof of Corollary 3.8.* By the factor property, the mp system  $(X, \mathcal{A}, \mu, T)$  inherits conservativity and ergodicity from  $(X^*, \mathcal{A}^*, \mu^*, T^*)$ .



Due to (1) and Lemmas 5.1 and 5.3,  $T_{Y^*}^*$  preserves  $\mu^*|_{Y^*} \ll \lambda^*|_{Y^*}$ , and is conservative ergodic w.r.t. this  $\sigma$ -finite measure. Note also that the extension is  $\tau$ -trivialising with base  $Y^*$ . By the second part of Theorem 3.1,  $T^\tau$  preserves  $\nu \ll \lambda|_Y$ , is conservative ergodic w.r.t.  $\nu$ , and the measures satisfy  $\mu = \tau \times_T \nu$ . Therefore the assertion follows from Theorem 3.7.  $\square$

**Acknowledgements.** The first author was supported by an NSERC Discovery Grant. Both authors wish to thank the Universities of Alberta and Vienna for their hospitality during mutual visits. They are also grateful to two anonymous referees whose comments helped improve the presentation of this work.

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