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Intermittent Synchronization in Finite-State Random Networks Under Markov Perturbations

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Abstract: By introducing extrinsic noise as well as intrinsic uncertainty into a network with stochastic events, this paper studies the dynamics of the resulting *Markov random network* and characterizes a novel phenomenon of intermittent synchronization and desynchronization that is due to an interplay of the two forms of randomness in the system. On a finite state space and in discrete time, the network allows for unperturbed (or "deterministic") randomness that represents the extrinsic noise but also for small intrinsic uncertainties modelled by a Markov perturbation. It is shown that if the deterministic random network is synchronized (resp., uniformly synchronized), then for almost all realizations of its extrinsic noise the stochastic trajectories of the perturbed network synchronize along almost all (resp., along all) time sequences after a certain time, with high probability. That is, both the probability of synchronization and the proportion of time spent in synchrony are arbitrarily close to one. Under smooth Markov perturbations, high-probability synchronization and low-probability desynchronization occur intermittently in time. If the perturbation is C^m ($m \geq 1$) in ε, where ε is a perturbation parameter, then the relative frequencies of synchronization with probability $1 - O(\varepsilon^\ell)$ and of desynchronization with probability $O(\varepsilon^\ell)$ can both be precisely described for $1 \leq \ell \leq m$ via an asymptotic expansion of the invariant distribution. Existence and uniqueness of invariant distributions are established, as well as their convergence as $\varepsilon \to 0$. An explicit asymptotic expansion is derived. Ergodicity of the extrinsic noise dynamics is seen to be crucial for the characterization of (de)synchronization sets and their respective relative frequencies. An example of a smooth Markov perturbation of a synchronized probabilistic Boolean network is provided to illustrate the intermittency between high-probability synchronization and low-probability desynchronization.

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1. Introduction

Physics, engineering, and sciences of complex systems and processes often encounter network dynamics that are subject to uncertainties from "individuals" in a population but also to random influences from the surrounding environment, referred to respectively as *intrinsic* and *extrinsic* noise. While the former is often due to internal complexities of the individuals one studies, the latter reflects the unpredictable world one lives in [9,15,16]. This paper proposes and develops the theory of *Markov random networks* as an appropriate conceptual framework that distinguishes intrinsic and extrinsic noise and incorporates them into a comprehensive dynamical theory. It has been speculated [8] that whereas extrinsic noise may cause "noise-induced synchronization", a familiar scenario in the context of random dynamical systems [5,6,10], intrinsic noise will drive synchronized individuals apart. The present work provides the first systematic analysis of this phenomenon. Adopting a simple discrete-time, finite-state framework for the stochastic dynamics on a network allows for a treatment that goes well beyond the standard approach available for non-autonomous or random dynamics on a more general space, and in particular enables an in-depth analysis of the intermittency between synchronization and desynchronization.

Let $S = \{s_1, \ldots, s_k\}$ be a finite set endowed with the discrete topology, and $\Theta := (\Omega, \mathcal{F}, \mu, \theta)$ an invertible metric dynamical system, that is, $(\Omega, \mathcal{F}, \mu)$ is a standard measure space with $\mu(\Omega) = 1$, and $\theta : \Omega \to \Omega$ is an invertible ergodic μ-preserving transformation. The view adopted herein is that Θ provides a model of the extrinsic noise. Call a stochastic process $\mathcal{X} = (X_n)_{n\in\mathbb{N}_0}$ with state space $S \times \Omega$ a *Markov random network* (MRN) if

(MRN1) $\mathcal{X}$ is measurable in distribution, that is, $\omega \mapsto \mathbb{P}\{X_n = (s_i, \theta^n\omega)|X_0 = (s_j, \omega)\}$ is $\mathcal{F}$-measurable for all $n \in \mathbb{N}_0$ and $i, j \in \{1, \ldots, k\}$;

(MRN2) $\mathcal{X}$ is stochastic over Θ, that is,

$$\sum\nolimits_{i=1}^{k} \mathbb{P}\{X_n = (s_i, \theta^{n-m}\omega)|X_m = (s_j, \omega)\} = 1 \quad \forall j \in \{1, \ldots, k\}$$

for all $n \geq m$ and μ-a.e. $\omega \in \Omega$;

(MRN3) $\mathcal{X}$ has the Markov property over Θ, that is,

$$\begin{aligned}&\mathbb{P}\{X_{n+1} = (s_{i_{n+1}}, \theta^{n+1}\omega) \mid X_n = (s_{i_n}, \theta^n\omega)\}\\&\quad = \mathbb{P}\{X_{n+1} = (s_{i_{n+1}}, \theta^{n+1}\omega) \mid X_0 = (s_{i_0}, \omega), \ldots, X_n = (s_{i_n}, \theta^n\omega)\}\end{aligned}$$

for all $n \in \mathbb{N}_0, i_0, \ldots, i_n, i_{n+1} \in \{1, \ldots, k\}$ and μ-a.e. $\omega \in \Omega$.

Given an MRN $\mathcal{X}$, let

$$p_{i,j}(n, \omega) = \mathbb{P}\{X_n = (s_i, \theta^n\omega)|X_0 = (s_j, \omega)\} \quad \forall i, j \in \{1, \ldots, k\}, n \in \mathbb{N}_0, \omega \in \Omega\,.$$

By properties (MRN 2) and (MRN 3), for every $n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$ the matrix

$$P_{\mathcal{X}}(n, \omega) := \big(p_{i,j}(n, \omega)\big)_{1\leq i,j\leq k} \in [0, 1]^{k\times k}$$

is (column-)stochastic, and the $2^{\mathbb{N}_0} \otimes \mathcal{F}$-measurable function $P_{\mathcal{X}} : \mathbb{N}_0 \times \Omega \to \mathbb{R}^{k\times k}$ has the *cocycle property*, that is, for μ-a.e. $\omega \in \Omega$,

$$P_{\mathcal{X}}(m + n, \omega) = P_{\mathcal{X}}(m, \theta^n\omega)P_{\mathcal{X}}(n, \omega) \quad \forall m, n \in \mathbb{N}_0\,.$$

Call $P_{\mathcal{X}}$, an example of a Markov cocycle, the *transition cocycle* of $\mathcal{X}$; see also Proposition 2.3 below. Furthermore, let

$$p_i(n, \omega) = \mathbb{P}\big\{X_n = (s_i, \theta^n \omega) | X_n \in S \times \{\theta^n \omega\}\big\} \quad \forall i \in \{1, \ldots, k\}, n \in \mathbb{N}_0, \omega \in \Omega\,.$$

With this, for every $n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$ the vector

$$p_{\mathcal{X}}(n, \omega) := \big(p_i(n, \omega)\big)_{1 \le i \le k} \in [0, 1]^k$$

can be thought of as the distribution of $\mathcal{X}$ on the fibre $S \times \{\theta^n \omega\}$; see also Proposition 2.3 below.

A special class of MRN are *deterministic random networks* (DRN) for which, by definition, the transition cocycle is *deterministic* in the sense that for every $n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$ each $p_{i,j}(n, \omega)$ equals either 0 or 1, that is, $P_{\mathcal{X}}(n, \omega) \in \{0, 1\}^{k \times k}$. Throughout this paper, for the sake of clarity, a DRN typically is denoted by $\mathcal{X}^0$, and its transition cocycle by $P^0 := P_{\mathcal{X}^0}$. Usage of the term "deterministic" emphasizes the absence of (internal) stochasticity between individual states in S. To put this terminology into context, note that every DRN $\mathcal{X}^0$ uniquely defines a so-called *discrete-time finite-state random dynamical system* (dtfs-RDS) on S over Θ. Specifically, for all $n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$ define maps $T_{\mathcal{X}^0} = T_{\mathcal{X}^0}(n, \omega) : S \to S$ such that for all $i, j \in \{1, \ldots, k\}$,

$$T_{\mathcal{X}^0}(n, \omega)(s_j) = s_i \quad \text{if and only if} \quad p^0_{i,j}(n, \omega) = 1. \tag{1.1}$$

It is easy to see that $T_{\mathcal{X}^0}$ inherits the cocycle property from P^0, that is,

$$T_{\mathcal{X}^0}(m + n, \omega) = T_{\mathcal{X}^0}(m, \theta^n \omega) \circ T_{\mathcal{X}^0}(n, \omega)$$

for all $m, n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$, and hence is a dtfs-RDS in the sense of [2]. Conversely, every dtfs-RDS on S over Θ induces a DRN. Prominent examples of DRN are, for instance, probabilistic Boolean networks that model gene regulations [12].

Adopting terminology from [6] say that a DRN $\mathcal{X}^0$ is *synchronized* if there exists an $\mathcal{F}$-measurable function $N : \Omega \to \mathbb{N}$ such that for all $i, j \in \{1, \ldots, k\}$ and μ-a.e. $\omega \in \Omega$,

$$T_{\mathcal{X}^0}(n, \omega)(s_i) = T_{\mathcal{X}^0}(n, \omega)(s_j) \quad \forall n \ge N(\omega)\,;$$

if N is constant μ-a.e. on Ω then $\mathcal{X}^0$ is *uniformly synchronized*. Synchronization of DRN is characterized in [6] in terms of the Lyapunov exponents of the associated (deterministic) transition cocycle P^0; in particular, it is shown that $\mathcal{X}^0$ is synchronized if and only if the Lyapunov exponent 0 of P^0 is simple.

The present paper focuses on MRN that are small perturbations of synchronized DRN. The main goal is to analyze the effect small perturbations have on synchronization. To be specific, given a DRN $\mathcal{X}^0$, call a family $\{\mathcal{X}^\varepsilon : \varepsilon \ge 0\}$ of MRN a *Markov perturbation* of $\mathcal{X}^0$ if for μ-a.e. $\omega \in \Omega$

$$|P_{\mathcal{X}^\varepsilon}(1, \omega) - P^0(1, \omega)| \le \varepsilon \quad \forall \varepsilon \ge 0\,;$$

here $|\cdot|$ denotes the norm on $\mathbb{R}^{k \times k}$ induced by the ℓ^1-norm on $\mathbb{R}^k$.

Markov perturbations of a DRN have a clear physical meaning: With small intrinsic noise added to the network (which itself allows only for extrinsic noise), transitions between states now occur with probabilities at most $O(\varepsilon)$ or at least $1 - O(\varepsilon)$, rather than being impossible or certain, respectively. Pertinent examples include probabilistic

Boolean networks with random gene perturbations, where each gene has a small probability of flipping its value — naturally, this yields a Markov perturbation of the original DRN. In general, existence, uniqueness, and attractiveness of an invariant distribution of an MRN all require certain monotonicity or Perron–Frobenius-type assumptions [3,4]. As the first main result of this work illustrates, however, these assumptions are satisfied automatically for any Markov perturbation of a synchronized DRN. In the statement, Σ_1^+ denotes the set of all probability distributions (or vectors) on S, and $e_1, \ldots, e_k$ is the canonical basis of $\mathbb{R}^k$; see Sect. 2 below for precise definitions of all technical terms.

Theorem A. *Let $\{\mathcal{X}^\varepsilon\}$ be a Markov perturbation of a synchronized DRN $\mathcal{X}^0$. Then, for every sufficiently small $\varepsilon \geq 0$, there exists an invariant distribution $p_\varepsilon : \Omega \to \Sigma_1^+$ of $\mathcal{X}^\varepsilon$, i.e., $P_{\mathcal{X}^\varepsilon}(1, \omega) p_\varepsilon(\omega) = p_\varepsilon(\theta\omega)$, with the following properties:*

(i) *p_ε is pull-back attracting for $\mathcal{X}^\varepsilon$, that is, for every $q \in \Sigma_1^+$ and for μ-a.e. $\omega \in \Omega$,*

$$\lim_{n\to\infty} |P_{\mathcal{X}^\varepsilon}(n, \theta^{-n}\omega)q - p_\varepsilon(\omega)| = 0\,;$$

(ii) *p_ε is forward attracting for $\mathcal{X}^\varepsilon$, that is, for every $q \in \Sigma_1^+$ and μ-a.e. $\omega \in \Omega$,*

$$\lim_{n\to\infty} |P_{\mathcal{X}^\varepsilon}(n, \omega)q - p_\varepsilon(\theta^n\omega)| = 0\,;$$

(iii) *p_ε is continuous at $\varepsilon = 0$, that is, there exists an $\mathcal{F}$-measurable function $J : \Omega \to \{1, \ldots, k\}$ such that for μ-a.e. $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0} p_\varepsilon(\omega) = e_{J(\omega)} = p_0(\omega)\,. \tag{1.2}$$

Moreover, if $\mathcal{X}^0$ is uniformly synchronized then the convergence in (1.2) is uniform μ-a.e. on Ω.

Theorem A is proved by analyzing Lyapunov exponents of the transition cocycle $P_{\mathcal{X}^\varepsilon}$. Specifically, it is the continuity of Lyapunov exponents of $P_{\mathcal{X}^\varepsilon}$ as $\varepsilon \to 0$ that yields the simplicity of the Lyapunov exponent 0 of $P_{\mathcal{X}^\varepsilon}$ for sufficiently small ε. The reader may want to recall that Lyapunov exponents of random cocycles in general are discontinuous with respect to generic perturbations [7,13]. As will be seen, the continuity of Lyapunov exponents in the context of Theorem A is due to a contraction property of the unperturbed cocycle P^0. The essence of Theorem A, then, is that this contraction property is preserved under Markov perturbations at the level of distributions. At the level of stochastic trajectories, the property is reflected by a high probability of synchronization. The second main result states that high-probability synchronization among stochastic trajectories prevails for large fractions of time.

Theorem B. *Let $\{\mathcal{X}^\varepsilon\}$ be a Markov perturbation of a synchronized DRN $\mathcal{X}^0$. Then, for every $\delta > 0$ there exist $\varepsilon_\delta > 0$ and $E_\delta : \Omega \to 2^{\mathbb{N}}$ such that for μ-a.e. $\omega \in \Omega$,*

(i) *$E_\delta(\omega)$ has large density, that is,*

$$\liminf_{n\to\infty} \frac{\#(E_\delta(\omega) \cap \{1, \ldots, n\})}{n} > 1 - \delta\,;$$

(ii) *any two independent copies $\mathcal{X}, \mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \leq \varepsilon < \varepsilon_\delta$ are very likely synchronized on $E_\delta(\omega)$, that is,*

$$\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} > 1 - \delta \quad \forall n \in E_\delta(\omega)\,.$$

Moreover, if $\mathcal{X}^0$ is uniformly synchronized then $E_\delta(\omega)$ is co-finite.

It is not hard to see that uniform synchronization of $\mathcal{X}^0$ is equivalent to P^0 being uniformly contracting on the hyper-plane $\Sigma_0 = \{v \in \mathbb{R}^k : \sum_{j=1}^k v_j = 0\}$, that is, with the appropriate constants $a > 0$ and $0 < \lambda < 1$,

$$|P^0(n,\omega)v| \le a\lambda^n |v| \quad \forall n \in \mathbb{N}_0, v \in \Sigma_0$$

for μ-a.e. $\omega \in \Omega$. In fact, $P^0(n,\omega)\Sigma_0 = \{0\}$ whenever $n \ge N(\omega)$. This uniform contraction property is a special form of uniform hyperbolicity, and as such is preserved under certain perturbations. It is clear, however, that a generic cocycle will not exhibit this property, and consequently a synchronized DRN will not in general be uniformly synchronized. (Sect. 6 provides an explicit example in this regard.) In the absence of uniform synchronization, the proof of Theorem B crucially depends on establishing connections between the probabilities of synchronization on the one hand and the asymptotic behaviour of the invariant distribution on the other hand. Once established, naturally these connections can be strengthened under additional smoothness assumptions. Specifically, say that a Markov perturbation $\{\mathcal{X}^\varepsilon\}$ is C^m $(m \ge 1)$ if $\varepsilon \mapsto P_{\mathcal{X}^\varepsilon}(1,\omega)$ is C^m on $[0,\varepsilon_0]$ for some $\varepsilon_0 > 0$ and μ-a.e. $\omega \in \Omega$. For C^m-Markov perturbations of a synchronized DRN this paper develops a Taylor formula for the invariant distribution p_ε of Theorem A, thus refining (1.2): With explicitly computable $\mathcal{F}$-measurable functions $q^{(\ell)} : \Omega \to \mathbb{R}^k$, $1 \le \ell \le m$, for μ-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(\omega) - e_{J(\omega)} - \sum_{\ell=1}^m \varepsilon^\ell q^{(\ell)}(\omega)/\ell!}{\varepsilon^m} = 0\,; \tag{1.3}$$

see Sect. 4 for details. Utilizing (1.3), it is possible to study quantitatively the scenarios of high-probability synchronization as well as low-probability desynchronization. In fact, these two scenarios coexist in an alternating fashion along most realizations of the extrinsic noise. To formulate this, the final main result of this work, consider the (possibly empty) set where $p_\varepsilon - e_J$ is completely degenerate, that is, let

$$\Omega_{\deg} = \{q^{(1)} = 0\} \cap \{q^{(2)} = 0\} \cap \ldots \cap \{q^{(m)} = 0\}\,. \tag{1.4}$$

Except when $\mu(\Omega_{\deg}) = 1$, where it is too degenerate to capture desynchronization, (1.3) enables a quantitative version of the coexistence claim as follows.

Theorem C. *Let $\{\mathcal{X}^\varepsilon\}$ be a C^m-Markov perturbation $(m \ge 1)$ of a synchronized DRN $\mathcal{X}^0$. Assume that $\mu(\Omega_{\deg}) < 1$. Then, with the appropriate $0 < a < 1$ and $\ell \in \{1,\ldots,m\}$, for every sufficiently small $\delta > 0$ there exist $\varepsilon_\delta > 0, b_\delta > 0, c_\delta > 0$, and $E_\delta, F_\delta : \Omega \to 2^{\mathbb{N}}$ such that for μ-a.e. $\omega \in \Omega$,*

(i) *$E_\delta(\omega)$, $F_\delta(\omega)$ are disjoint, have positive density, and together have large density, that is,*

$$\liminf_{n\to\infty} \frac{\#(E_\delta(\omega) \cap \{1,\ldots,n\})}{n} > a - \delta\,,$$
$$\liminf_{n\to\infty} \frac{\#(F_\delta(\omega) \cap \{1,\ldots,n\})}{n} > 1 - a - \delta\,;$$

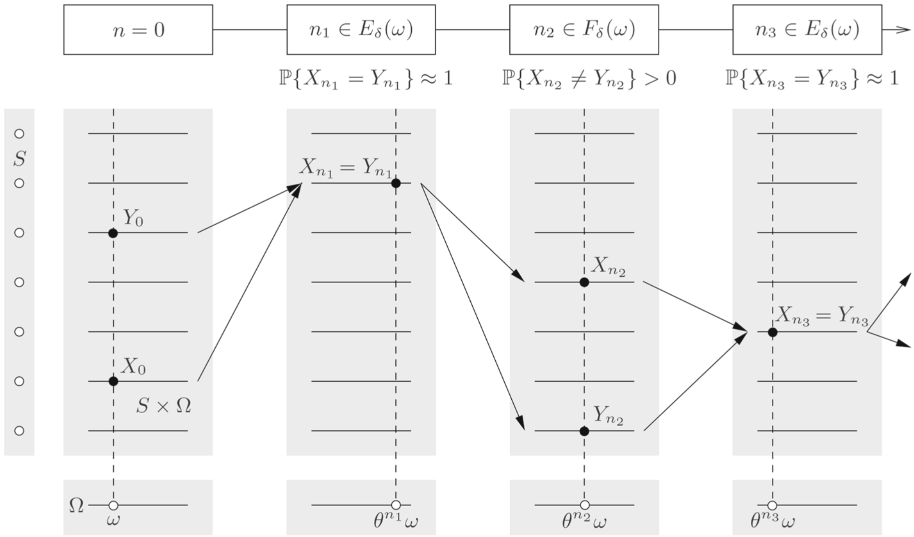

Fig. 1. By Theorem C, typical stochastic trajectories of a smooth Markov perturbation of a synchronized DRN exhibit alternating high-probability synchronization ($n \in E_\delta$) and low-probability desynchronization ($n \in F_\delta$)

(ii) *any two independent copies* $\mathcal{X}, \mathcal{Y}$ *of* $\mathcal{X}^\varepsilon$ *with* $0 \le \varepsilon < \varepsilon_\delta$ *are very likely synchronized on* $E_\delta(\omega)$ *but somewhat likely desynchronized on* $F_\delta(\omega)$*, that is,*

$$\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} \ge 1 - \varepsilon^\ell b_\delta \quad \forall n \in E_\delta(\omega)\,, \tag{1.5}$$

$$\mathbb{P}\big\{X_n \neq Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} \ge \varepsilon^\ell c_\delta \quad \forall n \in F_\delta(\omega). \tag{1.6}$$

As it turns out, the important quantities a and ℓ in Theorem C can be expressed rather explicitly in terms of $q^{(1)}, \dots, q^{(m)}$. For instance, if $\mu(\Omega_{\text{deg}}) > 0$ then simply $a = \mu(\Omega_{\text{deg}})$, and ℓ equals the smallest $1 \le i \le m$ for which $\mu(\{q^{(1)} = 0\} \cap \ldots \cap \{q^{(i)} = 0\}) = \mu(\Omega_{\text{deg}})$; see Sect. 5 below for details.

Note that the (very mild) non-degeneracy assumption $\mu(\Omega_{\text{deg}}) < 1$ is essential in Theorem C, as shown by the trivial example $\mathcal{X}^\varepsilon \equiv \mathcal{X}^0$ for which (ii) fails. Informally put, Theorem C asserts that along almost every extrinsic noise realization, for any two stochastic trajectories, high-probability synchronization (1.5) and low-probability desynchronization (1.6) occur along different time subsequences in an alternating way (see also Figure 1), and moreover, the combined relative frequencies of high-probability synchronization and low-probability desynchronization are arbitrarily close to 1. Thus the result rigorously confirms the speculation in [8] mentioned at the outset.

This paper is organized as follows. Section 2 reviews basic properties of MRN pertaining to the evolution of distributions as well as to Lyapunov exponents. Section 3 proves Theorem A by establishing the continuity of Lyapunov exponents for Markov perturbations of synchronized DRN, and also proves Theorem B by linking synchronization to invariant distributions. Section 4 studies the invariant distributions of smooth Markov perturbations, and derives the asymptotic formula (1.3), through which finer properties of p_ε can be investigated. Theorem C is proved in Sect. 5 utilizing (1.3). In Sect. 6 a concrete Markov perturbation of a probabilistic Boolean network illustrates the main concepts and results.

2. Basic Properties of MRN

This section establishes a few properties of Markov random networks that are instrumental in all that follows. To this end, first a modicum of linear algebra notation and terminology is reviewed.

Throughout, $k \geq 2$ is a positive integer, and $\mathbb{K} = \{1, \ldots, k\}$ for convenience; as needed, endow $\mathbb{K}$ with the topology, σ-algebra, and order inherited from $\mathbb{R}$. The canonical basis and identity map of $\mathbb{R}^k$ are denoted $e_1, \ldots, e_k$ and I_k, respectively, and $|\cdot|$ is the ℓ^1-norm on $\mathbb{R}^k$, that is, $|v| = \sum_{j\in\mathbb{K}} |v_j|$ for every $v = \sum_{j\in\mathbb{K}} v_j e_j \in \mathbb{R}^k$; as usual, $|\cdot|$ also denotes the induced norm on $\mathbb{R}^{k\times k}$, that is, $|A| = \max_{|v|=1}|Av| = \max_{j\in\mathbb{K}} \sum_{i\in\mathbb{K}} |a_{i,j}|$ for every $A = (a_{i,j}) \in \mathbb{R}^{k\times k}$. Given any $V \subset \mathbb{R}^k$ (or $V \subset \mathbb{R}^{k\times k}$), $a \in \mathbb{R}$, $u \in \mathbb{R}^k$ (or $u \in \mathbb{R}^{k\times k}$), and $A \in \mathbb{R}^{k\times k}$, write

$$aV = \{av : v \in V\}, \quad u + V = \{u + v : v \in V\}, \quad AV = \{Av : v \in V\}.$$

Also, for every $a \in \mathbb{R}$ let

$$\Sigma_a = \left\{v \in \mathbb{R}^k : \sum\nolimits_{j\in\mathbb{K}} v_j = a\right\}.$$

Plainly, Σ_0 is a $(k-1)$-dimensional linear subspace of $\mathbb{R}^k$, and $\Sigma_a = ae_1 + \Sigma_0$; also, $\min_{v\in\Sigma_a} |v| = |a|$, and if $u \in \Sigma_a$, $v \in \Sigma_b$ then $u \pm v \in \Sigma_{a\pm b}$. Furthermore, consider

$$\Sigma_a^+ = \left\{v \in \Sigma_a : v_j \geq 0 \ \forall j \in \mathbb{K}\right\},$$

and note that $\Sigma_a^+ = \varnothing$ if $a < 0$, $\Sigma_0^+ = \{0\}$, and $\Sigma_a^+ = a\Sigma_1^+$ if $a > 0$. In particular, Σ_1^+ may be identified with the set of all probability distributions on $\mathbb{K}$ as well as the standard simplex in $\mathbb{R}^k$, that is, the convex hull of $\{e_1, \ldots, e_k\}$.

For every $a \in \mathbb{R}$ let

$$\mathcal{M}_a = \left\{A \in \mathbb{R}^{k\times k} : \sum\nolimits_{i\in\mathbb{K}} a_{i,j} = a \ \forall j \in \mathbb{K}\right\}.$$

Plainly, $\mathcal{M}_0$ is a (k^2-k)-dimensional linear subspace of $\mathbb{R}^{k\times k}$, and if $A \in \mathcal{M}_a$, $B \in \mathcal{M}_b$ then $A \pm B \in \mathcal{M}_{a\pm b}$ and $AB \in \mathcal{M}_{ab}$. As is well known (and easy to check), the elements of $\mathcal{M}_a$ are characterized by simple invariance properties.

Proposition 2.1. *Let $A \in \mathbb{R}^{k\times k}$ Then*

(i) *$A \in \mathcal{M}_0$ if and only if $A\mathbb{R}^k \subset \Sigma_0$;*
(ii) *$A \in \mathcal{M}_a$ for some $a \in \mathbb{R}$ if and only if $A\Sigma_0 \subset \Sigma_0$;*
(iii) *$A \in \mathcal{M}_1$ if and only if $A\Sigma_a \subset \Sigma_a$ for some (and hence every) $a \neq 0$.*

Furthermore, consider

$$\mathcal{M}_a^+ = \{A \in \mathcal{M}_a : a_{i,j} \geq 0 \ \forall i, j \in \mathbb{K}\},$$

and note that $\mathcal{M}_a^+ = \varnothing$ if $a < 0$, $\mathcal{M}_0^+ = \{0\}$, and $\mathcal{M}_a^+ = a\mathcal{M}_1^+$ if $a > 0$. In particular, $\mathcal{M}_1^+$ is the set of all (column-)stochastic $k \times k$-matrices. A subclass of the latter particularly relevant for this work are the stochastic 0-1-matrices,

$$\mathcal{M}_{1,\text{det}}^+ = \left\{A \in \mathcal{M}_1^+ : a_{i,j} \in \{0, 1\} \ \forall i, j \in \mathbb{K}\right\},$$

informally referred to as *deterministic* stochastic matrices. Note that $\mathcal{M}_1^+$ and $\mathcal{M}_{1,\text{det}}^+$ are closed under matrix multiplication, just as $\mathcal{M}_0$ and $\mathcal{M}_1$ are. Not surprisingly, $\mathcal{M}_1^+$ and $\mathcal{M}_{1,\text{det}}^+$ also are characterized by simple invariance properties.

Proposition 2.2. *Let $A \in \mathbb{R}^{k\times k}$ Then*

(i) $A \in \mathcal{M}_1^+$ *if and only if* $A\Sigma_a^+ \subset \Sigma_a^+$ *for some (and hence every)* $a > 0$*, and in this case* $|A| = 1$*;*
(ii) $A \in \mathcal{M}_{1,\det}^+$ *if and only if* $A\{e_1, \dots, e_k\} \subset \{e_1, \dots, e_k\}$.

Recall from the Introduction that with every MRN one can associate the transition cocycle $P_{\mathcal{X}}$ and distribution $p_{\mathcal{X}}$. The following properties of $P_{\mathcal{X}}$ and $p_{\mathcal{X}}$ immediately follow from the definition of an MRN; property (ii) justifies usage of the term "cocycle".

Proposition 2.3. *Let $\mathcal{X}$ be an MRN with transition cocycle $P_{\mathcal{X}}$ and distribution $p_{\mathcal{X}}$. Then, for μ-a.e. $\omega \in \Omega$,*

(i) $P_{\mathcal{X}}(0, \omega) = I_k$ *and* $P_{\mathcal{X}}(n, \omega) \in \mathcal{M}_1^+$ *for all* $n \in \mathbb{N}_0$*;*
(ii) $P_{\mathcal{X}}$ *has the cocycle property*

$$P_{\mathcal{X}}(m+n, \omega) = P_{\mathcal{X}}(m, \theta^n\omega)P_{\mathcal{X}}(n, \omega) \quad \forall m, n \in \mathbb{N}_0\,,$$

and hence in particular

$$P_{\mathcal{X}}(n, \omega) = P_{\mathcal{X}}(1, \theta^{n-1}\omega)P_{\mathcal{X}}(1, \theta^{n-2}\omega)\cdots P_{\mathcal{X}}(1, \omega) \quad \forall n \in \mathbb{N}\,;$$

(iii) $p_{\mathcal{X}}(n, \omega) = P_{\mathcal{X}}(n, \omega)p_{\mathcal{X}}(0, \omega) \in \Sigma_1^+$ *for all* $n \in \mathbb{N}_0$.

Remark 2.4. (i) For the purpose of this paper, only the distributional structure of an MRN $\mathcal{X}$ matters. Whenever convenient, therefore, $P_{\mathcal{X}}$ and $p_{\mathcal{X}}$ may be replaced by $2^{\mathbb{N}_0} \otimes \mathcal{F}$-measurable functions $P' : \mathbb{N}_0 \times \Omega \to \mathcal{M}_1^+$ and $p' : \mathbb{N}_0 \times \Omega \to \Sigma_1^+$, respectively, such that

$$\mu\left(\bigcup\nolimits_{n\in\mathbb{N}_0}\{P'(n,\,\cdot) \neq P_{\mathcal{X}}(n,\,\cdot)\}\right) = \mu\left(\bigcup\nolimits_{n\in\mathbb{N}_0}\{p'(n,\,\cdot) \neq p_{\mathcal{X}}(n,\,\cdot)\}\right) = 0\,,$$

and the assertions of Proposition 2.3, with P', p' instead of $P_{\mathcal{X}}$, $p_{\mathcal{X}}$, hold for *all* $\omega \in \Omega$; cf. [2, sec.1.3.7].

(ii) One might call any $2^{\mathbb{N}_0} \otimes \mathcal{F}$-measurable function $P : \mathbb{N}_0 \times \Omega \to \mathcal{M}_1^+$ a *Markov cocycle* provided that it has the cocycle property. Except for convenience, however, nothing new is captured by this terminology: By Proposition 2.3, $P_{\mathcal{X}}$ is a Markov cocycle for every MRN $\mathcal{X}$, and conversely every Markov cocycle, e.g., the cocycle P' mentioned in (i), is the transition cocycle of an MRN.

To describe the long-time behaviour of an MRN $\mathcal{X}$, say that $\mathcal{X}$ *converges in distribution* if for μ-a.e. $\omega \in \Omega$,

$$\lim\nolimits_{n\to\infty}\left|P_{\mathcal{X}}(n, \omega)u - P_{\mathcal{X}}(n, \omega)v\right| = 0 \quad \forall u, v \in \Sigma_1^+\,,$$

or equivalently, $\lim_{n\to\infty}|P_{\mathcal{X}}(n, \omega)v| = 0$ for all $v \in \Sigma_0$. As the following lemma shows, in the special case of a DRN, convergence in distribution is the same as synchronization. Note that by Proposition 2.2 an MRN $\mathcal{X}$ is a DRN precisely if $P_{\mathcal{X}}(n, \omega) \in \mathcal{M}_{1,\det}^+$ for all $n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$.

Lemma 2.5. *Let $\mathcal{X}^0$ be a DRN. Then $\mathcal{X}^0$ converges in distribution if and only if $\mathcal{X}^0$ is synchronized.*

Proof. For the dtfs-RDS $T_{\mathcal{X}^0}$ associated with $\mathcal{X}^0$ by (1.1),

$$T_{\mathcal{X}^0}(n,\omega)(s_j) = s_i \quad \text{if and only if} \quad P^0(n,\omega)e_j = e_i \, .$$

Also, $P^0(n,\omega) \in \mathcal{M}^+_{1,\mathrm{det}}$, and so $|P^0(n,\omega)e_i - P^0(n,\omega)e_j| \in \{0,2\}$. Now, if $\mathcal{X}^0$ is synchronized then for all $i, j \in \mathbb{K}$ and μ-a.e. $\omega \in \Omega$,

$$P^0(n,\omega)e_i = P^0(n,\omega)e_j \quad \forall n \geq N(\omega) \, .$$

Since every $v \in \Sigma_0$ is a linear combination of $\{e_i - e_j : i, j \in \mathbb{K}\}$, also $P^0(n,\omega)v = 0$ for all $n \geq N(\omega)$. Conversely, if $\mathcal{X}^0$ converges in distribution then for μ-a.e. $\omega \in \Omega$ the set

$$L(\omega) := \{n \in \mathbb{N}_0 : P^0(n,\omega)e_i = P^0(n,\omega)e_j \ \forall i, j \in \mathbb{K}\}$$

is co-finite, and hence with $N(\omega) = \inf L(\omega)$ the DRN $\mathcal{X}^0$ is synchronized. □

As will be seen next, even in the more general case of an arbitrary MRN it is possible to characterize convergence in distribution. To this end, given an MRN $\mathcal{X}$ with transition cocycle $P_{\mathcal{X}}$, call an $\mathcal{F}$-measurable function $p : \Omega \to \Sigma^+_1$ an *invariant distribution* of $\mathcal{X}$ if for μ-a.e. $\omega \in \Omega$,

$$P_{\mathcal{X}}(n,\omega)p(\omega) = p(\theta^n \omega) \quad \forall n \in \mathbb{N}_0 \, .$$

Lemma 2.6. *Let $\mathcal{X}$ be an MRN with transition cocycle $P_{\mathcal{X}}$. Then the following statements are equivalent:*

(i) *$\mathcal{X}$ converges in distribution;*
(ii) *there exists an invariant distribution p of $\mathcal{X}$ such that for every $q \in \Sigma^+_1$ and μ-a.e. $\omega \in \Omega$,*

$$\lim_{n\to\infty} |P_{\mathcal{X}}(n,\theta^{-n}\omega)q - p(\omega)| = 0$$

("p is pull-back attracting");
(iii) *there exists an invariant distribution p of $\mathcal{X}$ such that for every $q \in \Sigma^+_1$ and μ-a.e. $\omega \in \Omega$,*

$$\lim_{n\to\infty} |P_{\mathcal{X}}(n,\omega)q - p(\theta^n\omega)| = 0 \tag{2.1}$$

("p is forward attracting").

Moreover, the invariant distributions p in (ii) *and* (iii) *are uniquely determined and coincide μ-a.e. on Ω.*

Proof. To prove (i)⇒(ii), assume $\mathcal{X}$ converges in distribution. For μ-a.e. $\omega \in \Omega$ and every $\ell \in \mathbb{N}$ pick $N_\ell(\omega) \in \mathbb{N}$ such that

$$|P_{\mathcal{X}}(n,\omega)e_i - P_{\mathcal{X}}(n,\omega)e_j| < \frac{1}{\ell} \quad \forall i, j \in \mathbb{K}, n \geq N_\ell(\omega) \, .$$

Also pick $m_\ell \in \mathbb{N}$ with $m_\ell \geq \ell$ and $\mu(\{N_\ell \leq m_\ell\}) > 0$; assume w.l.o.g. that $(m_\ell)_{\ell\in\mathbb{N}}$ is increasing. By Poincaré recurrence, it can be assumed that $\theta^{-n}\omega \in \{N_\ell \leq m_\ell\}$ for infinitely many n; in particular, pick $M_\ell(\omega) > m_\ell + M_{\ell-1}(\omega) + M_{\ell-1}(\theta^{-1}\omega)$, where

$M_0 := 0$, such that $\theta^{-M_\ell(\omega)}\omega \in \{N_\ell \le m_\ell\}$ for μ-a.e. $\omega \in \Omega$ and all $\ell \in \mathbb{N}$. With this, consider the compact set

$$C_\ell(\omega) := P_{\mathcal{X}}(M_\ell(\omega), \theta^{-M_\ell(\omega)}\omega)\Sigma_1^+ \ \subset \ \Sigma_1^+ .$$

Since $N_\ell(\theta^{-M_\ell(\omega)}\omega) \le m_\ell$ whereas $M_\ell(\omega) > N_\ell(\omega)$, clearly $\operatorname{diam} C_\ell(\omega) < 1/\ell$. Moreover, by the cocycle property, for every $n \ge M_\ell(\omega)$,

$$P_{\mathcal{X}}(n, \theta^{-n}\omega)\Sigma_1^+ = P_{\mathcal{X}}(M_\ell(\omega), \theta^{-M_\ell(\omega)}\omega)P_{\mathcal{X}}(n - M_\ell(\omega), \theta^{-n}\omega)\Sigma_1^+ \subset C_\ell(\omega) , \tag{2.2}$$

and hence in particular $C_{\ell+1}(\omega) \subset C_\ell(\omega)$. It follows that for μ-a.e. $\omega \in \Omega$ there exists a unique $p(\omega) \in \Sigma_1^+$ with $\{p(\omega)\} = \bigcap_{\ell\in\mathbb{N}} C_\ell(\omega)$. Clearly, $p : \Omega \to \Sigma_1^+$ can be chosen to be $\mathcal{F}$-measurable. Furthermore, for μ-a.e. $\omega \in \Omega$ and all $\ell \in \mathbb{N}$, since $M_{\ell+1}(\theta\omega) \ge M_\ell(\omega) + 1$,

$$\begin{aligned} p(\theta\omega) \in C_{\ell+1}(\theta\omega) &= P_{\mathcal{X}}(1, \omega)P_{\mathcal{X}}(M_{\ell+1}(\theta\omega) - 1, \theta^{1-M_{\ell+1}(\theta\omega)}\omega)\Sigma_1^+ \\ &\subset P_{\mathcal{X}}(1, \omega)C_\ell(\omega) . \end{aligned}$$

Since ℓ has been arbitrary, $p(\theta\omega) = P_{\mathcal{X}}(1, \omega)p(\omega)$, that is, p is an invariant distribution of $\mathcal{X}$. Finally, given $q \in \Sigma_1^+$, deduce from (2.2) that $P_{\mathcal{X}}(n, \theta^{-n}\omega)q \in C_\ell(\omega)$ for μ-a.e. $\omega \in \Omega$ and $n \ge M_\ell(\omega)$. Thus

$$|P_{\mathcal{X}}(n, \theta^{-n}\omega)q - p(\omega)| \le \frac{2}{\ell} \quad \forall n \ge M_\ell(\omega) ,$$

and consequently $\lim_{n\to\infty} |P_{\mathcal{X}}(n, \theta^{-n}\omega)q - p(\omega)| = 0$.

To prove (ii)⇒(iii), let $p : \Omega \to \Sigma_1^+$ be an invariant distribution as in (ii). For μ-a.e. $\omega \in \Omega$ and every $\ell \in \mathbb{N}$ pick $N'_\ell(\omega) \in \mathbb{N}$ such that

$$|P_{\mathcal{X}}(n, \theta^{-n}\omega)e_j - p(\omega)| < \frac{1}{\ell} \quad \forall j \in \mathbb{K}, n \ge N'_\ell(\omega) .$$

Similarly to above, pick $m'_\ell \in \mathbb{N}$ with $m'_\ell \ge \ell$ and $\mu(\{N'_\ell \le m'_\ell\}) > 0$ such that $(m'_\ell)_{\ell\in\mathbb{N}}$ is increasing. Again by Poincaré recurrence, one can choose $M'_\ell(\omega) \ge m'_\ell$ with $\theta^{M'_\ell(\omega)}\omega \in \{N'_\ell \le m'_\ell\}$, and consequently

$$|P_{\mathcal{X}}(M'_\ell(\omega), \omega)e_j - p(\theta^{M'_\ell(\omega)}\omega)| < \frac{1}{\ell} \quad \forall j \in \mathbb{K} .$$

For every $q \in \Sigma_1^+$ and μ-a.e. $\omega \in \Omega$, therefore, since p is invariant,

$$\begin{aligned} &|P_{\mathcal{X}}(n, \omega)q - p(\theta^n\omega)| \\ &\quad = \left|P_{\mathcal{X}}(n - M'_\ell(\omega), \theta^{M'_\ell(\omega)}\omega)\big(P_{\mathcal{X}}(M'_\ell(\omega), \omega)q - p(\theta^{M'_\ell(\omega)}\omega)\big)\right| \\ &\quad \le |P_{\mathcal{X}}(M'_\ell(\omega), \omega)q - p(\theta^{M'_\ell(\omega)}\omega)| < \frac{1}{\ell} , \end{aligned}$$

provided that $n \ge M'_\ell(\omega)$. Since $\ell \in \mathbb{N}$ has been arbitrary, this proves (2.1).

Finally, to see that (iii)⇒(i) simply choose $q = e_i$ and $q = e_j$, respectively, and observe that for μ-a.e. $\omega \in \Omega$,

$$\begin{aligned} |P_{\mathcal{X}}(n, \omega)e_i - P_{\mathcal{X}}(n, \omega)e_j| &\le |P_{\mathcal{X}}(n, \omega)e_i - p(\theta^n\omega)| \\ &\quad + |P_{\mathcal{X}}(n, \omega)e_j - p(\theta^n\omega)| \to 0 \quad \text{as } n \to \infty , \end{aligned}$$

showing that $\mathcal{X}$ converges in distribution.

The assertion regarding the μ-a.e. uniqueness of the invariant distribution is obvious (and justifies usage of the common symbol p). □

Note that for a DRN $\mathcal{X}^0$ every invariant distribution p of $\mathcal{X}^0$ is concentrated on a single state, that is, $p(\omega) = e_{J(\omega)}$ for a unique $J(\omega) \in \mathbb{K}$. Lemmas 2.5 and 2.6 together therefore have the following corollary.

Proposition 2.7. *Let $\mathcal{X}^0$ be a synchronized DRN. Then there exist $\mathcal{F}$-measurable functions $J : \Omega \to \mathbb{K}$ and $N^\pm : \Omega \to \mathbb{N}$ such that for μ-a.e. $\omega \in \Omega$,*

$$P^0(n, \omega)e_{J(\omega)} = e_{J(\theta^n\omega)} \quad \forall n \in \mathbb{N}_0 ,$$

and also, for every $j \in \mathbb{K}$,

$$P^0(n, \omega)e_j = e_{J(\theta^n\omega)} \quad \forall n \geq N^+(\omega) ,$$
$$P^0(n, \theta^{-n}\omega)e_j = e_{J(\omega)} \quad \forall n \geq N^-(\omega) .$$

Any function J as in Proposition 2.7 henceforth is referred to as a *synchronization index* of the DRN $\mathcal{X}^0$, and N^+, N^- are a *forward* and a *pull-back synchronization time*, respectively. Note that J is determined uniquely μ-a.e. on Ω, whereas $N^\pm$ clearly are not. Also, it is not hard to see that $N^\pm$ can be assumed constant μ-a.e. on Ω whenever $\mathcal{X}^0$ is uniformly synchronized.

Remark 2.8. As pointed out by the referee, it would be possible to *define*, for μ-a.e. $\omega \in \Omega$,

$$N^+(\omega) = \min\{n \in \mathbb{N} : P^0(n, \omega)e_j = e_{J(\theta^n\omega)} \ \forall j \in \mathbb{K}\} ,$$

and similarly for N^-. However, to emphasize that crucial parts of this work, notably expression (4.1) below, are independent of the specific choice for (forward or pull-back) synchronization times, no such definitions are adopted. Unless explicitly stated otherwise, synchronization times $N^\pm$ can be arbitrary as long as they comply with Proposition 2.7; see also Proposition 6.1 below.

The remainder of this section reviews a few pertinent facts regarding Lyapunov exponents. Given an MRN $\mathcal{X}$, for every $\omega \in \Omega$ and $v \in \mathbb{R}^k$ let

$$\lambda_{\mathcal{X}}(\omega, v) = \limsup_{n\to\infty} \frac{\log |P_{\mathcal{X}}(n, \omega)v|}{n} , \tag{2.3}$$

with the convention that $\log 0 := -\infty$. As a consequence of the classical Multiplicative Ergodic Theorem [2,11], for μ-a.e. $\omega \in \Omega$ the lim sup in (2.3) actually is a limit, and $\lambda_{\mathcal{X}}(\omega, \cdot)$ attains at most k different real values which are constant μ-a.e. on Ω. In fact, since $|P_{\mathcal{X}}(n, \omega)| = 1$ and $P_{\mathcal{X}}(n, \omega)\Sigma_a \subset \Sigma_a$ for every a by Propositions 2.1 and 2.3, a bit more can be said.

Proposition 2.9. *Let $\mathcal{X}$ be an MRN. Then, for μ-a.e. $\omega \in \Omega$ and every $v \in \mathbb{R}^k$,*

$$\lambda_{\mathcal{X}}(\omega, v) \leq 0 , \tag{2.4}$$

and equality holds in (2.4) whenever $v \notin \Sigma_0$; in particular, 0 is a Lyapunov exponent of $\mathcal{X}$.

In the special case of a DRN, the Lyapunov exponents $\lambda^0 := \lambda_{\mathcal{X}^0}$ can equal only $-\infty$ or 0; as it turns out, they also characterize synchronization.

Proposition 2.10. [6, Thm.A] *Let $\mathcal{X}^0$ be a DRN. Then $\mathcal{X}^0$ is synchronized if and only if for μ-a.e. $\omega \in \Omega$,*

$$\lambda^0(\omega, v) = \begin{cases} -\infty & \text{if } v \in \Sigma_0 , \\ 0 & \text{if } v \notin \Sigma_0 . \end{cases}$$

By noting that $e_i - e_j \in \Sigma_0$ for all $i, j \in \mathbb{K}$, a simple sufficient condition for convergence in distribution of an MRN follows immediately.

Proposition 2.11. *Let $\mathcal{X}$ be an MRN. If $\lambda_{\mathcal{X}}(\omega, v) < 0$ for μ-a.e. $\omega \in \Omega$ and all $v \in \Sigma_0$ then $\mathcal{X}$ converges in distribution.*

Remark 2.12. In [4], a random Perron–Frobenius theorem is established for positive (that is, strongly monotone) cocycles under log-integrability conditions, which in turn yields convergence in distribution. In general, however, the transition cocycle $P_{\mathcal{X}}$ of an MRN $\mathcal{X}$ neither is positive (due to possible zero entries) nor does it satisfy the log-integrability conditions.

3. Convergence in Distribution Under Markov Perturbations

This section establishes two of the main results of this work, Theorems A and B. The proofs depend on three simple observations, of which the continuity of Lyapunov exponents under Markov perturbations (Lemma 3.3) may be of independent interest. Convergence in distribution, as well as continuity of the invariant distribution, follow directly from the continuity of Lyapunov exponents. First, observe that the cocycle property and an induction argument immediately yield

Proposition 3.1. *Let $\{\mathcal{X}^\varepsilon\}$ be a Markov perturbation of a DRN $\mathcal{X}^0$. Then, for μ-a.e. $\omega \in \Omega$,*

$$|P_{\mathcal{X}^\varepsilon}(n, \omega) - P^0(n, \omega)| \le n\varepsilon \quad \forall \varepsilon \ge 0, n \in \mathbb{N}_0 .$$

Next, given any synchronized DRN $\mathcal{X}^0$, recall from Proposition 2.7 the notion of forward synchronization time $N^+ : \Omega \to \mathbb{N}$. Along a typical realization of the extrinsic noise the value of N^+ reasonably often is not too large.

Lemma 3.2. *Let $\mathcal{X}^0$ be a synchronized DRN, and N^+ a forward synchronization time. Let $m \in \mathbb{N}$ be such that $\mu(\{N^+ \le m\}) > 0$. Then, for μ-a.e. $\omega \in \Omega$ there exists a sequence $(n_\ell)_{\ell \in \mathbb{N}}$ such that $n_{\ell+1} - n_\ell \ge m$ and $N^+(\theta^{n_\ell}\omega) \le m$ for all $\ell \in \mathbb{N}$, as well as* $\limsup_{\ell \to \infty} \ell / n_\ell \ge \mu(\{N^+ \le m\})/m$.

Proof. For μ-a.e. $\omega \in \Omega$, the Birkhoff ergodic theorem yields

$$\lim_{n \to \infty} \frac{\#(\{k \in \mathbb{N} : N^+(\theta^k \omega) \le m\} \cap \{1, \ldots, n\})}{n} = \mu(\{N^+ \le m\}) ,$$

and hence for at least one $1 \le i \le m$, possibly depending on ω,

$$\limsup_{n \to \infty} \frac{\#(\{k \in \mathbb{N} : N^+(\theta^k \omega) \le m\} \cap (i + m\mathbb{N}) \cap \{1, \ldots, n\})}{n} \ge \frac{\mu(\{N^+ \le m\})}{m} .$$

In particular, the set $\{k \in \mathbb{N} : N^+(\theta^k \omega) \le m\} \cap (i + m\mathbb{N})$ is infinite, and writing it as $\{n_\ell : \ell \in \mathbb{N}\}$ with $n_1 < n_2 < \ldots$ yields a sequence (n_ℓ) that has all the asserted properties. □

The final preliminary observation establishes the continuity of all Lyapunov exponents of MRN that are Markov perturbations of a synchronized DRN. As pointed out in the Introduction, synchronization of the unperturbed DRN is crucial for this result.

Lemma 3.3. *Let $\{\mathcal{X}^\varepsilon\}$ be a Markov perturbation of a synchronized DRN $\mathcal{X}^0$. Then, for every sufficiently small $\varepsilon \geq 0$, for μ-a.e. $\omega \in \Omega$ and every $v \in \mathbb{R}^k$, the Lyapunov exponents of $\mathcal{X}^\varepsilon$ satisfy*

$$\lambda_{\mathcal{X}^\varepsilon}(\omega, v) < 0 \text{ if } v \in \Sigma_0 , \quad \lambda_{\mathcal{X}^\varepsilon}(\omega, v) = 0 \text{ if } v \notin \Sigma_0 . \tag{3.1}$$

Moreover, $\lambda_{\mathcal{X}^\varepsilon}$ is continuous at $\varepsilon = 0$, that is, $\lim_{\varepsilon \to 0} \lambda_{\mathcal{X}^\varepsilon}(\omega, v) = \lambda^0(\omega, v)$.

Proof. By Propositions 2.9 and 2.10 all assertions are correct in case $v \notin \Sigma_0$, so henceforth assume $v \in \Sigma_0$. Since $\mathcal{X}^0$ is synchronized, pick $m \in \mathbb{N}$ with $\mu(\{N^+ \leq m\}) > 0$, and fix $\varepsilon < 1/m$. For μ-a.e. $\omega \in \Omega$, let $(n_\ell)_{\ell \in \mathbb{N}}$ be as in Lemma 3.2. Then $P^0(m, \theta^{n_\ell}\omega)v = 0$, and since $n_{\ell+1} - n_\ell \geq m$,

$$\begin{aligned} &|P_{\mathcal{X}^\varepsilon}(n_{\ell+1} - n_\ell, \theta^{n_\ell}\omega)v| \\ &\quad \leq |P_{\mathcal{X}^\varepsilon}(m, \theta^{n_\ell}\omega)v| = \left|\left(P_{\mathcal{X}^\varepsilon}(m, \theta^{n_\ell}\omega) - P^0(m, \theta^{n_\ell}\omega)\right)v\right| \leq m\varepsilon|v| , \end{aligned} \tag{3.2}$$

by Proposition 3.1. Recall from Proposition 2.1 that $P_{\mathcal{X}^\varepsilon}(n, \omega)\Sigma_0 \subset \Sigma_0$ for all n, so (3.2) can be iterated,

$$\begin{aligned} |P_{\mathcal{X}^\varepsilon}(n_\ell, \omega)v| &= |P_{\mathcal{X}^\varepsilon}(n_\ell - n_{\ell-1}, \theta^{n_{\ell-1}}\omega)P_{\mathcal{X}^\varepsilon}(n_{\ell-1} - n_{\ell-2}, \theta^{n_{\ell-2}}\omega) \\ &\qquad \cdots P_{\mathcal{X}^\varepsilon}(n_2 - n_1, \theta^{n_1}\omega)P_{\mathcal{X}^\varepsilon}(n, \omega)v| \\ &\leq (m\varepsilon)^{\ell-1}|v| , \end{aligned}$$

which in turn yields, utilizing Lemma 3.2,

$$\begin{aligned} \lambda_{\mathcal{X}^\varepsilon}(\omega, v) &= \lim_{n\to\infty} \frac{\log|P_{\mathcal{X}^\varepsilon}(n, \omega)v|}{n} \leq \liminf_{\ell\to\infty} \frac{(\ell - 1)\log(m\varepsilon) + \log|v|}{n_\ell} \\ &= \log(m\varepsilon) \limsup_{\ell\to\infty} \frac{\ell - 1}{n_\ell} \leq \frac{\log(m\varepsilon)\mu(\{N^+ \leq m\})}{m} < 0 . \end{aligned}$$

Thus (3.1) holds, and letting $\varepsilon \to 0$ shows that $\lim_{\varepsilon\to 0} \lambda_{\mathcal{X}^\varepsilon}(\omega, v) = -\infty$ whenever $v \in \Sigma_0$. Proposition 2.10 then concludes the proof. □

The scene is now set for a rather straightforward

Proof of Theorem A. To prove the existence of an invariant distribution, as well as (i) and (ii), simply observe that Lemma 3.3 and Proposition 2.11 together imply that $\mathcal{X}^\varepsilon$ converges in distribution for all sufficiently small $\varepsilon > 0$. By Lemma 2.6, there exists a unique invariant distribution of $\mathcal{X}^\varepsilon$, denoted p_ε, that is both pull-back and forward attracting.

To establish (iii), that is, the continuity of $\varepsilon \mapsto p_\varepsilon$ at $\varepsilon = 0$, recall from Proposition 2.7 that $P^0(N^-(\omega), \theta^{-N^-(\omega)}\omega)v = e_{J(\omega)}$ for μ-a.e. $\omega \in \Omega$ and every $v \in \Sigma_1^+$, where N^- denotes a pull-back synchronization time. By the invariance of p_ε and Proposition 3.1,

$$\begin{aligned} &|p_\varepsilon(\omega) - e_{J(\omega)}| \\ &\quad = |P_{\mathcal{X}^\varepsilon}(N^-(\omega), \theta^{-N^-(\omega)}\omega)p_\varepsilon(\theta^{-N^-(\omega)}\omega) - e_{J(\omega)}| \\ &\quad = \left|\left(P_{\mathcal{X}^\varepsilon}(N^-(\omega), \theta^{-N^-(\omega)}\omega) - P^0(N^-(\omega), \theta^{-N^-(\omega)}\omega)\right)p_\varepsilon(\theta^{-N^-(\omega)}\omega)\right| \\ &\quad \leq N^-(\omega)\varepsilon , \end{aligned} \tag{3.3}$$

and hence clearly $\lim_{\varepsilon\to 0} p_\varepsilon(\omega) = e_{J(\omega)}$ for μ-a.e. $\omega \in \Omega$.

Finally, if $\mathcal{X}^0$ is uniformly synchronized then $N^-(\omega) \le N$ for μ-a.e. $\omega \in \Omega$ and the appropriate $N \in \mathbb{N}$, so (3.3) shows that the convergence in (1.2) is uniform on a set of full μ-measure. □

Note that, informally put, Theorem A provides a description, at the level of (invariant) distributions, of what synchronization looks like for a sufficiently small Markov perturbation of a synchronized DRN. In order to rephrase this description at the level of stochastic trajectories (Theorem B), the following simple linear algebra observation is helpful.

Proposition 3.4. *Let $v \in \Sigma_1^+$. Then $v_j = 1 - |v - e_j|/2$ for every $j \in \mathbb{K}$.*

Proof of Theorem B. Fix $\varepsilon_0 > 0$ so small that all conclusions in Theorem A hold whenever $0 \le \varepsilon < \varepsilon_0$. In particular, therefore, given $\delta > 0$, for μ-a.e. $\omega \in \Omega$ there exists $N(\omega) \in \mathbb{N}$ and $f(\omega, \delta) > 0$ such that

$$|p_{\mathcal{X}^\varepsilon}(n, \omega) - p_\varepsilon(\theta^n\omega)| < \frac{\delta}{2} \;\forall n \ge N(\omega) \quad \text{and} \quad |p_\varepsilon(\omega) - e_{J(\omega)}| < \frac{\delta}{2} \;\forall \varepsilon < f(\omega, \delta)\,.$$

Pick $0 < \varepsilon_\delta \le \varepsilon_0$ so small that $\mu(\{f(\cdot, \delta) \ge \varepsilon_\delta\}) > 1-\delta$. Letting $\Omega_\delta = \{f(\cdot, \delta) \ge \varepsilon_\delta\}$ and $E_\delta(\omega) = \{n \ge N(\omega) : \theta^n\omega \in \Omega_\delta\}$ for convenience, by the Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{\#(E_\delta(\omega) \cap \{1, \dots, n\})}{n} = \mu(\Omega_\delta) > 1 - \delta\,,$$

which proves (i). Furthermore, if $0 < \varepsilon < \varepsilon_\delta$ and $n \in E_\delta(\omega)$ then $|p_{\mathcal{X}^\varepsilon}(n, \omega) - e_{J(\theta^n\omega)}| < \delta$ by the triangle inequality, and

$$\begin{aligned}
&\mathbb{P}\big\{X_n = (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0 \in S \times \{\omega\}\big\} \\
&\quad = p_{\mathcal{X}^\varepsilon}(n, \omega)_{J(\theta^n\omega)} = 1 - \frac{|p_{\mathcal{X}^\varepsilon}(n, \omega) - e_{J(\theta^n\omega)}|}{2} \\
&\quad > 1 - \frac{\delta}{2}\,,
\end{aligned}$$

where the second equality is due to Proposition 3.4. Thus, if $\mathcal{X}, \mathcal{Y}$ are two independent copies of $\mathcal{X}^\varepsilon$, with $0 < \varepsilon < \varepsilon_\delta$, then for all $n \in E_\delta(\omega)$,

$$\begin{aligned}
&\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} \\
&\quad \ge \mathbb{P}\big\{X_n = Y_n = (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0, Y_0 \in S \times \{\omega\}\big\} > \left(1 - \frac{\delta}{2}\right)^2 \\
&\quad > 1 - \delta\,,
\end{aligned}$$

which proves (ii).

Finally, if $\mathcal{X}^0$ is uniformly synchronized then, by Theorem A it can be assumed that $f(\omega, \delta)$ is independent of ω. But then $\mu(\Omega_\delta) = 1$, indeed $\mu\left(\bigcap_{n\ge 0} \theta^{-n}\Omega_\delta\right) = 1$, and so for μ-a.e. $\omega \in \Omega$ the set $E_\delta(\omega) = \{n \ge N(\omega)\}$ is co-finite. □

4. Invariant Distributions for Smooth Markov Perturbations

This section establishes an asymptotic expansion for the invariant distribution of a C^m-Markov perturbation $\{\mathcal{X}^\varepsilon\}$, with $m \geq 1$, of a synchronized DRN $\mathcal{X}^0$. Remember that due to the "deterministic" nature of $\mathcal{X}^0$, at all times each state leads to precisely one subsequent state. By contrast, due to the presence of intrinsic noise, states of $\mathcal{X}^\varepsilon$ typically have a small but positive probability of leading to more than one subsequent state. As may be expected, these deviations can be described in terms of the derivatives of $\varepsilon \mapsto P_{\mathcal{X}^\varepsilon}$ at $\varepsilon = 0$. To do this explicitly, for every $0 \leq \ell \leq m$ let

$$P^{(\ell)}_{\mathcal{X}^\varepsilon}(n, \omega) = \left.\frac{\mathrm{d}^\ell}{\mathrm{d}\varepsilon^\ell} P_{\mathcal{X}^\varepsilon}(n, \omega)\right|_{\varepsilon=0} ;$$

note that $P^{(0)}_{\mathcal{X}^\varepsilon}(n, \omega) = P^0(n, \omega) \in \mathcal{M}^+_{1,\mathrm{det}}$, and clearly $P^{(\ell)}_{\mathcal{X}^\varepsilon}$ is $2^{\mathbb{N}_0} \otimes \mathcal{F}$-measurable, with $P^{(\ell)}_{\mathcal{X}^\varepsilon}(n, \omega) \in \mathcal{M}_0$ for all $1 \leq \ell \leq m$, $n \in \mathbb{N}_0$, and μ-a.e. $\omega \in \Omega$. Moreover, for every $n \in \mathbb{N}_0$ and μ-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \frac{P_{\mathcal{X}^\varepsilon}(n, \omega) - P^0(n, \omega) - \sum_{\ell=1}^m \varepsilon^\ell P^{(\ell)}_{\mathcal{X}^\varepsilon}(n, \omega)/\ell!}{\varepsilon^m} = 0 .$$

Throughout this section, assume that the DRN $\mathcal{X}^0$ is synchronized. By Proposition 2.7, $\mathcal{X}^0$ admits an (essentially unique) synchronization index J as well as a pull-back synchronization time N^-. The following simple observation will be used several times below.

Proposition 4.1. *Let $\mathcal{X}^0$ be a synchronized DRN, and N^- a pull-back synchronization time. Then, for μ-a.e. $\omega \in \Omega$,*

$$P^0(N^-(\omega), \theta^{-N^-(\omega)}\omega)v = 0 \quad \forall v \in \Sigma_0 .$$

Utilizing J and N^-, as well as $P^{(\ell)}_{\mathcal{X}^\varepsilon}$, define $\mathcal{F}$-measurable functions $q^{(0)}, q^{(1)}, \ldots, q^{(m)} : \Omega \to \mathbb{R}^k$ as

$$q^{(0)}(\omega) = e_{J(\omega)} ,$$

and inductively for $1 \leq \ell \leq m$,

$$q^{(\ell)}(\omega) = \sum_{i=0}^{\ell-1} \binom{\ell}{i} P^{(\ell-i)}_{\mathcal{X}^\varepsilon}(N^-(\omega), \theta^{-N^-(\omega)}\omega) q^{(i)}(\theta^{-N^-(\omega)}\omega) . \tag{4.1}$$

Note that $q^{(0)} \in \Sigma_1^+$, whereas clearly $q^{(\ell)}(\omega) \in \Sigma_0$ for all $1 \leq \ell \leq m$ and μ-a.e. $\omega \in \Omega$; in particular,

$$q^{(1)}(\omega) = P^{(1)}_{\mathcal{X}^\varepsilon}(N^-(\omega), \theta^{-N^-(\omega)}\omega) e_{J(\theta^{-N^-(\omega)}\omega)} .$$

The main result of this section, Lemma 4.2 below, provides a Taylor formula approximately expressing the invariant distribution of $\mathcal{X}^\varepsilon$ in terms of the quantities $q^{(\ell)}$, $1 \leq \ell \leq m$. As the attentive reader will have noticed, by definition these quantities depend on the choice of N^-. However, it is readily seen that choosing a different N^- yields the same $q^{(1)}, \ldots, q^{(m)}$ for μ-a.e. $\omega \in \Omega$.

Lemma 4.2. *Let $\{\mathcal{X}^\varepsilon\}$ be a C^m-Markov perturbation ($m \geq 1$) of a synchronized DRN $\mathcal{X}^0$. For every sufficiently small $\varepsilon \geq 0$ let p_ε be the invariant distribution of $\mathcal{X}^\varepsilon$, as in Theorem A. Then, for μ-a.e. $\omega \in \Omega$,*

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(\omega) - e_{J(\omega)} - \sum_{\ell=1}^{m} \varepsilon^\ell q^{(\ell)}(\omega)/\ell!}{\varepsilon^m} = 0\,. \tag{4.2}$$

Proof. With N^- denoting any pull-back synchronization time of $\mathcal{X}^0$, write $\omega^- := \theta^{-N^-(\omega)}\omega$ for convenience. To establish (4.2), it will be shown that for every $0 \leq k \leq m$ and μ-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(\omega) - e_{J(\omega)} - \sum_{\ell=1}^{k} \varepsilon^\ell q^{(\ell)}(\omega)/\ell!}{\varepsilon^k} = 0\,. \tag{4.3}$$

By Theorem A(ii), clearly (4.3) is correct for $k = 0$, with an "empty" sum understood to equal 0, as usual. Assume that (4.3) is correct for some $0 \leq k < m$. Then

$$\begin{aligned} P_{\mathcal{X}^\varepsilon}(n,\omega) = P^0(n,\omega) + \sum\nolimits_{\ell=1}^{k} \frac{\varepsilon^\ell}{\ell!} P^{(\ell)}_{\mathcal{X}^\varepsilon}(n,\omega) + \frac{\varepsilon^{k+1}}{(k+1)!} P^{(k+1)}_{\mathcal{X}^\varepsilon}(n,\omega) \\ + \varepsilon^{k+1} R_\varepsilon(n,\omega)\,, \end{aligned}$$

with the appropriate $R_\varepsilon(n,\omega) \in \mathcal{M}_0$ and $\lim_{\varepsilon\to 0} R_\varepsilon(n,\omega) = 0$, as well as

$$p_\varepsilon(\omega) = e_{J(\omega)} + \sum\nolimits_{\ell=1}^{k} \frac{\varepsilon^\ell}{\ell!} q^{(\ell)}(\omega) + \varepsilon^k r_\varepsilon(\omega)\,,$$

with the appropriate $r_\varepsilon(\omega) \in \Sigma_0$ and $\lim_{\varepsilon\to 0} r_\varepsilon(\omega) = 0$. Thus, by the invariance of p_ε and e_J, Proposition 4.1, and the definition of $q^{(\ell)}$,

$$\begin{aligned} p_\varepsilon(\omega) - e_{J(\omega)} &= P_{\mathcal{X}^\varepsilon}(N^-(\omega),\omega^-) p_\varepsilon(\omega^-) - e_{J(\omega)} \\ &= \big(P_{\mathcal{X}^\varepsilon}(N^-(\omega),\omega^-) - P^0(N^-(\omega),\omega^-)\big) p_\varepsilon(\omega^-) \\ &= \Bigg(\sum\nolimits_{\ell=1}^{k} \frac{\varepsilon^\ell}{\ell!} P^{(\ell)}_{\mathcal{X}^\varepsilon}(N^-(\omega),\omega^-) + \frac{\varepsilon^{k+1}}{(k+1)!} P^{(k+1)}_{\mathcal{X}^\varepsilon}(N^-(\omega),\omega^-) \\ &\qquad + \varepsilon^{k+1} R_\varepsilon(N^-(\omega),\omega^-)\Bigg) \\ &\qquad \cdot \Bigg(e_{J(\omega^-)} + \sum\nolimits_{\ell=1}^{k} \frac{\varepsilon^\ell}{\ell!} q^{(\ell)}(\omega^-) + \varepsilon^k r_\varepsilon(\omega^-)\Bigg) \\ &= \sum\nolimits_{\ell=1}^{k} \sum\nolimits_{i=0}^{\ell-1} \frac{\varepsilon^\ell}{i!(\ell-i)!} P^{(\ell-i)}_{\mathcal{X}^\varepsilon}(N^-(\omega),\omega^-) q^{(i)}(\omega^-) \\ &\qquad + \varepsilon^{k+1} \sum\nolimits_{i=0}^{k} \frac{1}{i!(k+1-i)!} P^{(k+1-i)}_{\mathcal{X}^\varepsilon}(N^-(\omega),\omega^-) q^{(i)}(\omega^-) \\ &\qquad + \varepsilon^{k+1} R'_\varepsilon(n,\omega) \\ &= \sum\nolimits_{\ell=1}^{k+1} \frac{\varepsilon^\ell}{\ell!} q^{(\ell)}(\omega) + \varepsilon^{k+1} R'_\varepsilon(n,\omega)\,, \end{aligned}$$

with the appropriate $R'_\varepsilon(n,\omega) \in \mathcal{M}_0$ and $\lim_{\varepsilon\to 0} R'_\varepsilon(n,\omega) = 0$. Thus (4.3) is correct with k replaced by $k+1$, and induction establishes (4.2). □

Remark 4.3. For a homogeneous Markov chain, a series expansion (or updating formula) for invariant distributions under (regular as well as singular) perturbations has been derived in [1]. In this context, the transition cocycle $P_{\mathcal{X}^\varepsilon}$ may be viewed as a "random" version of a Markov chain, with the Markov perturbation as a regular perturbation. Accordingly, Lemma 4.2 may be viewed as a random version of the series expansion in [1] under regular perturbations.

The remainder of this section develops a simple condition guaranteeing that the invariant distribution p_ε of a Markov perturbation $\{\mathcal{X}^\varepsilon\}$ is degenerate at $\varepsilon = 0$ in that $q^{(1)} = 0$ in (4.2). In this, a crucial role is played by the quantity

$$d(\omega) := P^{(1)}_{\mathcal{X}^\varepsilon}(1, \omega) e_{J(\omega)} ,$$

which may be thought of as the first-order (one-step) *probability dissipation* under the Markov perturbation. Notice that d is $\mathcal{F}$-measurable, and $d(\omega) \in \Sigma_0$ for μ-a.e. $\omega \in \Omega$. Moreover, except for the $J(\theta\omega)$-th component, all components of $d(\omega)$ are non-negative, that is, $d(\omega)_j \geq 0$ for every $j \in \mathbb{K} \backslash \{J(\theta\omega)\}$, and hence $d(\omega)_{J(\theta\omega)} \leq 0$. From the cocycle properties of $P_{\mathcal{X}^\varepsilon}$ and P^0, it is readily deduced that for all $n \in \mathbb{N}$ and μ-a.e. $\omega \in \Omega$,

$$P^{(1)}_{\mathcal{X}^\varepsilon}(n, \omega) = \sum_{\ell=0}^{n-1} P^0(n-1-\ell, \theta^{\ell+1}\omega) P^{(1)}_{\mathcal{X}^\varepsilon}(1, \theta^\ell \omega) P^0(\ell, \omega) , \tag{4.4}$$

and consequently $q^{(1)}$ can be written neatly in terms of d, thus revealing its pull-back nature,

$$\begin{aligned} q^{(1)}(\omega) &= \sum_{\ell=0}^{N^-(\omega)-1} P^0(N^-(\omega)-1-\ell, \theta^{1+\ell-N^-(\omega)}\omega) d(\theta^{\ell-N^-(\omega)}\omega) \\ &= \sum_{\ell=0}^{N^-(\omega)-1} P^0(\ell, \theta^{-\ell}\omega) d(\theta^{-\ell-1}\omega) . \end{aligned} \tag{4.5}$$

In order to trace the dissipation of probability, for μ-a.e. $\omega \in \Omega$ consider the subset $S_\bullet$ of S given by

$$S_\bullet(\omega) = \{s_j \in S : P^0(1, \omega) e_j = e_{J(\theta\omega)}\} .$$

Thus $S_\bullet(\omega)$ contains precisely those states that lead to the synchronized state s_J in a single step. By invariance, $s_{J(\omega)} \in S_\bullet(\omega)$. Also, consider the subset of Ω given by

$$\Omega_\bullet = \{\omega \in \Omega : d(\omega)_j = 0 \ \forall s_j \notin S_\bullet(\theta\omega)\} \in \mathcal{F} .$$

Thus, for $\omega \in \Omega_\bullet$ first-order probability dissipation occurs only to states that immediately lead to the synchronized state; see also Figure 2. Intuitively, it is plausible that if first-order probability dissipation is thus restricted for all times up to the (finite) synchronization time of $\mathcal{X}^0$ then p_ε differs from the single-state invariant distribution e_J of $\mathcal{X}^0$ only by higher orders of ε, that is, $q^{(1)} = 0$.

Lemma 4.4. *Let $\{\mathcal{X}^\varepsilon\}$ be a C^m-Markov perturbation ($m \geq 1$) of a synchronized DRN $\mathcal{X}^0$, with pull-back synchronization time N^-. Then, for μ-a.e. $\omega \in \Omega$, the following hold:*

(i) *If $d(\theta^{-1}\omega) \neq 0$ then $q^{(1)}(\omega) \neq 0$;*
(ii) *If $d(\theta^{-1}\omega) = 0$ and $\omega \in \bigcap_{\ell=1}^{N^-(\omega)-1} \theta^{\ell+1}\Omega_\bullet$ then $q^{(1)}(\omega) = 0$.*

Proof. Notice first that each of the $N^-(\omega)$ terms in (4.5) lies in Σ_0, with all components non-negative, except possibly for the $J(\omega)$-th component. Thus $q^{(1)} = 0$ if and only if $P^0(\ell, \theta^{-\ell}\omega)d(\theta^{-\ell-1}\omega) = 0$ for all $\ell = 0, 1, \ldots, N^-(\omega) - 1$. With this, on the one hand, if $d(\theta^{-1}\omega) \neq 0$ then $q^{(1)}(\omega) \neq 0$, which proves (i). On the other hand, if $\omega \in \theta^{\ell+1}\Omega_\bullet$ for some $1 \le \ell \le N^-(\omega) - 1$ then $P^0(1, \theta^{-\ell}\omega)d(\theta^{-\ell-1}\omega)$ lies in Σ_0, with all components non-negative, except for the $J(\theta^{1-\ell}\omega)$-th component, for which

$$\begin{aligned}
&P^0(1, \theta^{-\ell}\omega)d(\theta^{-\ell-1}\omega)_{J(\theta^{1-\ell}\omega)} \\
&= \sum\nolimits_{j=1}^{k} P^0(1, \theta^{-\ell}\omega)_{J(\theta^{1-\ell}\omega), j} d(\theta^{-\ell-1}\omega)_j \\
&= \sum\nolimits_{j\in S_\bullet(\theta^{-\ell}\omega)} P^0(1, \theta^{-\ell}\omega)_{J(\theta^{1-\ell}\omega), j} d(\theta^{-\ell-1}\omega)_j \\
&= \sum\nolimits_{j\in S_\bullet(\theta^{-\ell}\omega)} d(\theta^{-\ell-1}\omega)_j = \sum\nolimits_{j=1}^{k} d(\theta^{-\ell-1}\omega)_j = 0\,,
\end{aligned}$$

where the second and fourth equality are due to $\omega \in \theta^{\ell+1}\Omega_\bullet$. Thus $P^0(1, \theta^{-\ell}\omega)d(\theta^{-\ell-1}\omega) = 0$, and by the cocycle property $P^0(\ell, \theta^{-\ell}\omega)d(\theta^{-\ell-1}\omega) = 0$ as well, that is, $q^{(1)}(\omega) = 0$, which proves (ii). □

Remark 4.5. In a spirit similar to Lemma 4.4(ii), one might derive conditions for higher-order degeneracy of p_ε at $\varepsilon = 0$, that is, for $q^{(1)} = \ldots = q^{(\ell)} = 0$ for some $2 \le \ell \le m$. However, since the pertinent analogues of (4.1) and (4.5) are considerably more cumbersome in this case, such conditions likely are of limited practical use.

5. Alternating Patterns of Synchronization and Desynchronization

This section is devoted to the proof of Theorem C. As with Theorem B, the main idea is to link the behaviour of the invariant distribution p_ε as $\varepsilon \to 0$ to synchronization at the level of trajectories. Utilizing the asymptotic expansion (4.2) enables a refinement of Theorem B, via quantitative descriptions of high-probability synchronization as well as low-probability desynchronization. For the sake of clarity, these two scenarios are first addressed in two separate lemmas; a combination of both results then yields Theorem C. Throughout, assume that $\{\mathcal{X}^\varepsilon\}$ is a C^m-Markov perturbation ($m \ge 1$) of a synchronized DRN $\mathcal{X}^0$. For convenience, for every $1 \le \ell \le m$ let

$$\Omega^{(\ell)} = \{q^{(1)} = 0\} \cap \ldots \cap \{q^{(\ell)} = 0\} \in \mathcal{F}\,; \tag{5.1}$$

note that $\Omega^{(0)} := \Omega \supset \Omega^{(1)} \supset \ldots \supset \Omega^{(m)} = \Omega_{\deg}$, with $\Omega_{\deg}$ considered already in (1.4).

Lemma 5.1. *Assume that* $\mu\big(\Omega^{(\ell)}\big) > 0$ *for some* $1 \le \ell \le m$. *Then, for every* $0 < \delta < \mu\big(\Omega^{(\ell)}\big)$ *there exist* $\varepsilon_\delta > 0$, $b_\delta : [0, \varepsilon_\delta) \to \mathbb{R}$ *with* $b_\delta(\varepsilon) \ge 0$ *and* $\lim_{\varepsilon\to 0} b_\delta(\varepsilon) = 0$, *and* $E_\delta : \Omega \to 2^{\mathbb{N}}$ *such that for* μ*-a.e.* $\omega \in \Omega$,

(i) $\liminf_{n\to\infty} \dfrac{\#(E_\delta(\omega) \cap \{1, ..., n\})}{n} > \mu(\Omega^{(\ell)}) - \delta$;
(ii) *for any two independent copies* $\mathcal{X}$, $\mathcal{Y}$ *of* $\mathcal{X}^\varepsilon$ *with* $0 \le \varepsilon < \varepsilon_\delta$,

$$\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} \ge 1 - \varepsilon^\ell b_\delta(\varepsilon) \quad \forall n \in E_\delta(\omega)\,.$$

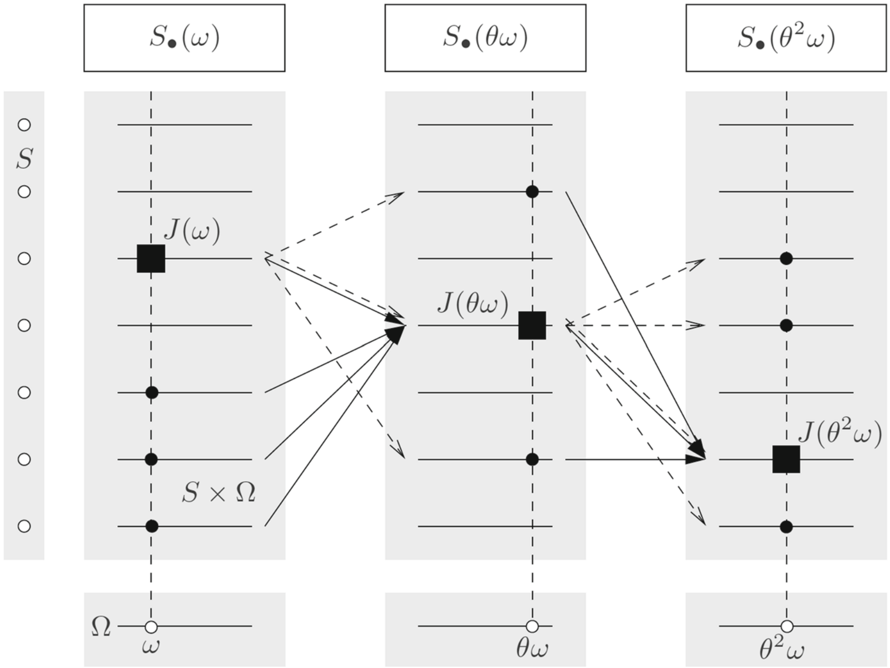

Fig. 2. For $\omega \in \Omega_\bullet$, first-order probability dissipation (dashed arrows) under a smooth Markov perturbation occurs only to states that immediately lead to the synchronized state (black squares) of the synchronized DRN (solid arrows)

Proof. Fix $\varepsilon_0 > 0$ so small that all conclusions of Theorem A hold whenever $0 \le \varepsilon < \varepsilon_0$, and pick any $0 < \delta < \mu\big(\Omega^{(\ell)}\big)$. By Lemma 4.2, for μ-a.e. $\omega \in \Omega^{(\ell)}$,

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(\omega) - e_{J(\omega)}}{\varepsilon^\ell} = 0\,, \tag{5.2}$$

so by Egorov's theorem there exists $C_\delta \subset \Omega^{(\ell)}$ with $\mu(C_\delta) > \mu\big(\Omega^{(\ell)}\big) - \delta$ such that the convergence in (5.2) is uniform on C_δ. Letting $b_\delta(\varepsilon) = \varepsilon + \sup_{\omega\in C_\delta} |p_\varepsilon(\omega) - e_{J(\omega)}|/\varepsilon^\ell$ for $0 < \varepsilon < \varepsilon_0$, note that $\lim_{\varepsilon\to 0} b_\delta(\varepsilon) = 0 =: b_\delta(0)$, and

$$|p_\varepsilon(\omega) - e_{J(\omega)}| \le \varepsilon^\ell (b_\delta(\varepsilon) - \varepsilon) \quad \forall 0 \le \varepsilon < \varepsilon_0, \omega \in C_\delta\,.$$

Also, by Theorem A(i) there exists $N(\omega) \in \mathbb{N}$ such that

$$|p_{\mathcal{X}^\varepsilon}(n,\omega) - p_\varepsilon(\theta^n\omega)| \le \varepsilon^{\ell+1} \quad \forall n \ge N(\omega)\,.$$

Let $E_\delta(\omega) = \{n \ge N(\omega) : \theta^n\omega \in C_\delta\}$. Then, for μ-a.e. $\omega \in \Omega$ the set $E_\delta(\omega) \subset \mathbb{N}$ satisfies (i), and for every $n \in E_\delta(\omega)$,

$$\begin{aligned}\mathbb{P}\big\{X_n = (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0 \in S \times \{\omega\}\big\} &= p_{\mathcal{X}^\varepsilon}(n,\omega)_{J(\theta^n\omega)} \\ &= 1 - \frac{|p_{\mathcal{X}^\varepsilon}(n,\omega) - e_{J(\theta^n\omega)}|}{2} \ge 1 - \frac{\varepsilon^\ell b_\delta(\varepsilon)}{2}\,,\end{aligned}$$

provided that $0 \le \varepsilon < \varepsilon_0$. Consequently, for any two independent copies $\mathcal{X}$, $\mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \le \varepsilon < \varepsilon_0$,

$$\begin{aligned}\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} &\ge \mathbb{P}\big\{X_n = Y_n = (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0, Y_0 \in S \times \{\omega\}\big\} \\ &\ge \left(1 - \frac{\varepsilon^\ell b_\delta(\varepsilon)}{2}\right)^2 \\ &\ge 1 - \varepsilon^\ell b_\delta(\varepsilon)\end{aligned}$$

for every $n \in E_\delta(\omega)$, which proves (ii) with $\varepsilon_\delta = \varepsilon_0$. □

Remark 5.2. As seen in the above proof, in Lemma 5.1 one may stipulate that ε_δ be independent of δ: Simply take $\varepsilon_\delta = \varepsilon_0$ with ε_0 so small that all conclusions of Theorem A hold whenever $0 \le \varepsilon < \varepsilon_0$. The wording of Lemma 5.1 has been chosen for consistency with Lemma 5.3 below, as well as Theorems B and C, where ε_δ does depend on δ.

Lemma 5.3. *Assume that $\mu\big(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\big) > 0$ for some $1 \le \ell \le m$. Then, for every $0 < \delta < \mu\big(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\big)$ there exist $\varepsilon_\delta > 0$, $c_\delta > 0$, and $F_\delta : \Omega \to 2^{\mathbb{N}}$ such that for μ-a.e. $\omega \in \Omega$,*

(i) $\liminf_{n\to\infty} \dfrac{\#(F_\delta(\omega) \cap \{1, \dots, n\})}{n} > \mu\big(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\big) - \delta$;

(ii) *for any two independent copies $\mathcal{X}$, $\mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \le \varepsilon < \varepsilon_\delta$,*

$$\mathbb{P}\big\{X_n \ne Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} \ge \varepsilon^\ell c_\delta \quad \forall n \in F_\delta(\omega)\,.$$

Proof. Fix $\varepsilon_0 > 0$ as in the proof of Lemma 5.1, and pick any $0 < \delta < \mu\big(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\big)$. By Lemma 4.2, for μ-a.e. $\omega \in \Omega^{(\ell-1)} \setminus \Omega^{(\ell)}$,

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon(\omega) - e_{J(\omega)} - \varepsilon^\ell q^{(\ell)}(\omega)/\ell!}{\varepsilon^\ell} = 0\,, \tag{5.3}$$

so one can choose $D_\delta \subset \Omega^{(\ell-1)} \setminus \Omega^{(\ell)}$ with $\mu(D_\delta) > \mu\big(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\big) - \delta$ such that the convergence in (5.3) is uniform on D_δ, but also $0 < c_\delta \le 1/6$ such that $2c_\delta \le |q^{(\ell)}(\omega)/\ell!| \le 2/c_\delta$ for all $\omega \in D_\delta$. Let $0 < \varepsilon_\delta \le \min\{\varepsilon_0, c_\delta^2\}$ be so small that

$$\left| p_\varepsilon(\omega) - e_{J(\omega)} - \frac{\varepsilon^\ell q^{(\ell)}(\omega)}{\ell!} \right| < \varepsilon^\ell c_\delta^2 \quad \forall 0 \le \varepsilon < \varepsilon_\delta, \omega \in D_\delta\,.$$

As in the proof of Lemma 5.1, there exists $N(\omega) \in \mathbb{N}$ with

$$|p_{\mathcal{X}^\varepsilon}(n, \omega) - p_\varepsilon(\theta^n\omega)| \le \varepsilon^{\ell+1} \quad \forall n \ge N(\omega)\,.$$

Then $F_\delta(\omega) := \{n \ge N(\omega) : \theta^n\omega \in D_\delta\} \subset \mathbb{N}$ satisfies (i) for μ-a.e. $\omega \in \Omega$. Moreover, for every $n \in F_\delta(\omega)$,

$$\begin{aligned}&\mathbb{P}\big\{X_n = (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0 \in S \times \{\omega\}\big\} \\ &= p_{\mathcal{X}^\varepsilon}(n, \omega)_{J(\theta^n\omega)} = 1 - \frac{|p_{\mathcal{X}^\varepsilon}(n, \omega) - e_{J(\theta^n\omega)}|}{2} \\ &\ge 1 - \frac{\varepsilon^{\ell+1} + c_\delta^2\varepsilon^\ell + 2\varepsilon^\ell/c_\delta}{2} \\ &\ge 1 - \varepsilon^\ell(c_\delta^2 + 1/c_\delta)\,,\end{aligned}$$

provided that $0 \le \varepsilon < \varepsilon_\delta$, but also

$$\begin{aligned}\mathbb{P}\bigl\{X_n \neq (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0 \in S \times \{\omega\}\bigr\} \\ = \frac{|p_{\mathcal{X}^\varepsilon}(n,\omega) - e_{J(\theta^n\omega)}|}{2} \ge c_\delta \varepsilon^\ell - \frac{\varepsilon^{\ell+1} + c_\delta^2 \varepsilon^\ell}{2} \\ \ge \varepsilon^\ell c_\delta (1 - c_\delta)\,,\end{aligned}$$

and consequently, for any two independent copies $\mathcal{X}, \mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \le \varepsilon < \varepsilon_\delta$,

$$\begin{aligned}\mathbb{P}\bigl\{X_n \neq Y_n | X_0, Y_0 \in S \times \{\omega\}\bigr\} &\ge 2\bigl(1 - \varepsilon^\ell (c_\delta^2 + 1/c_\delta)\bigr)\varepsilon^\ell c_\delta (1 - c_\delta) \\ &= \varepsilon^\ell \bigl(2c_\delta(1 - c_\delta) - 2\varepsilon^\ell (1 - c_\delta)(c_\delta^3 + 1)\bigr) \\ &\ge 2\varepsilon^\ell c_\delta (1 - c_\delta)(1 - 2c_\delta) \\ &\ge \varepsilon^\ell c_\delta\,,\end{aligned}$$

which establishes (ii). □

By combining Lemmas 5.1 and 5.3, it is now straightforward to provide a

Proof of Theorem C. Deduce from

$$0 < 1 - \mu(\Omega_{\deg}) = \sum\nolimits_{i=1}^m \mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr)$$

that $\mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr) > 0$ for at least one $1 \le i \le m$, and so

$$L := \bigl\{1 \le i \le m : \mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr) > 0\bigr\} \neq \varnothing\,. \tag{5.4}$$

Letting $\ell = \max L$ for convenience, clearly $\mu\bigl(\Omega^{(\ell-1)}\bigr) \ge \mu\bigl(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\bigr) > 0$ and $\mu\bigl(\Omega^{(\ell)}\bigr) = \mu(\Omega_{\deg}) < 1$, as well as $\sum_{i\in L} \mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr) = 1 - \mu\bigl(\Omega^{(\ell)}\bigr)$.

Assume first that $\mu\bigl(\Omega^{(\ell)}\bigr) > 0$, and hence $0 < \mu\bigl(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\bigr) < 1$. By Lemma 5.3, for every $i \in L$ and $0 < \delta < \mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr)$ there exist $0 < \varepsilon_{\delta,i} \le 1$, $c_{\delta,i} > 0$, and $F_{\delta,i} : \Omega \to 2^{\mathbb{N}}$ such that for μ-a.e. $\omega \in \Omega$,

$$\liminf_{n\to\infty} \frac{\#(F_{\delta,i}(\omega) \cap \{1,\ldots,n\})}{n} > \mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr) - \frac{\delta}{m}\,,$$

and for any two independent copies $\mathcal{X}, \mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \le \varepsilon < \varepsilon_{\delta,i}$,

$$\mathbb{P}\bigl\{X_n \neq Y_n | X_0, Y_0 \in S \times \{\omega\}\bigr\} \ge \varepsilon^i c_{\delta,i} \ge \varepsilon^\ell c_{\delta,i} \quad \forall n \in F_{\delta,i}(\omega)\,.$$

The sets $F_{\delta,i}$ are disjoint by construction. With this, for every $0 < \delta < \min_{i\in L} \mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr)$, let $\varepsilon'_\delta = \min_{i\in L} \varepsilon_{\delta,i} > 0$ and $c_\delta = \min_{i\in L} c_{\delta,i} > 0$, as well as $F_\delta(\omega) = \bigcup_{i\in L} F_{\delta,i}(\omega)$. Then,

$$\begin{aligned}\liminf_{n\to\infty} \frac{\#(F_\delta(\omega) \cap \{1,\ldots,n\})}{n} \\ > \sum\nolimits_{i\in L} \left(\mu\bigl(\Omega^{(i-1)} \setminus \Omega^{(i)}\bigr) - \frac{\delta}{m}\right) \ge 1 - \mu\bigl(\Omega^{(\ell)}\bigr) - \delta\,.\end{aligned}$$

By Lemma 5.1, for every $0 < \delta < \mu\big(\Omega^{(\ell)}\big)$ there exist $\varepsilon_{\delta,0} > 0$, $b_{\delta,0}(\varepsilon) \ge 0$ with $\lim_{\varepsilon\to 0} b_{\delta,0}(\varepsilon) = 0 = b_{\delta,0}(0)$, and $E_\delta : \Omega \to 2^{\mathbb{N}}$ such that for μ-a.e. $\omega \in \Omega$,

$$\liminf_{n\to\infty} \frac{\#(E_\delta(\omega)\cap\{1,\ldots,n\})}{n} > \mu\big(\Omega^{(\ell)}\big) - \delta\,,$$

and for any two independent copies $\mathcal{X}, \mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \le \varepsilon < \varepsilon_{\delta,0}$,

$$\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} \ge 1 - \varepsilon^\ell b_{\delta,0}(\varepsilon) \quad \forall n \in E_\delta(\omega)\,.$$

Again, $E_\delta(\omega)\cap F_\delta(\omega) = \varnothing$ by construction, provided that $\delta < \min\big\{\mu\big(\Omega^{(\ell)}\big), \mu\big(\Omega^{(i-1)}\setminus \Omega^{(i)}\big) : i \in L\big\}$. Pick $\varepsilon'_{\delta,0} > 0$ so small that $b_{\delta,0}(\varepsilon) \le c_\delta$ whenever $0 \le \varepsilon < \varepsilon'_{\delta,0}$. With $\varepsilon_\delta := \min\{\varepsilon'_\delta, \varepsilon_{\delta,0}, \varepsilon'_{\delta,0}\} > 0$ and $b_\delta := c_\delta$, therefore, all assertions of the theorem are correct with $a = \mu\big(\Omega^{(\ell)}\big)$.

Next assume that $\mu\big(\Omega^{(\ell)}\big) = 0$ but $0 < \mu\big(\Omega^{(\ell-1)}\big) = \mu\big(\Omega^{(\ell-1)} \setminus \Omega^{(\ell)}\big) < 1$. Then $\#L \ge 2$ and $\ell \ge 2$, hence the same argument as before applies, and Theorem C holds, with ℓ replaced by $\ell - 1 \ge 1$ and $a = \mu\big(\Omega^{(\ell-1)}\big)$.

It remains to consider the case of $\mu\big(\Omega^{(\ell)}\big) = 0$ but $\mu\big(\Omega^{(\ell-1)}\big) = \mu\big(\Omega^{(\ell-1)}\setminus\Omega^{(\ell)}\big) = 1$, or equivalently $\#L = 1$. As in the proof of Lemma 5.3, for every $0 < \delta < 1$ one can choose $C_\delta \subset \Omega$ with $\mu(C_\delta) > 1 - \delta$ and $0 < c_\delta \le 1/6$ such that $2c_\delta \le |q^{(\ell)}(\omega)/\ell!| \le 2/c_\delta$ for all $\omega \in C_\delta$, and with $0 < \varepsilon_\delta \le \min\{\varepsilon_0, c_\delta^2\}$ sufficiently small,

$$\left| p_\varepsilon(\omega) - e_{J(\omega)} - \frac{\varepsilon^\ell q^{(\ell)}(\omega)}{\ell!} \right| \le \varepsilon^\ell c_\delta^2 \quad \forall 0 \le \varepsilon < \varepsilon_\delta, \omega \in C_\delta\,.$$

Also, there exists $N(\omega) \in \mathbb{N}$ with

$$|p_{\mathcal{X}^\varepsilon}(n,\omega) - p_\varepsilon(\theta^n\omega)| \le \varepsilon^{\ell+1} \quad \forall n \ge N(\omega)\,.$$

The set $D_\delta(\omega) := \{n \ge N(\omega) : \theta^n\omega \in C_\delta\} \subset \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \frac{\#(D_\delta(\omega)\cap\{1,\ldots,n\})}{n} = \mu(C_\delta) > 1 - \delta$$

for μ-a.e. $\omega \in \Omega$, and for every $n \in D_\delta(\omega)$,

$$\mathbb{P}\big\{X_n = (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0 \in S \times \{\omega\}\big\} \ge 1 - \varepsilon^\ell(c_\delta^2 + 1/c_\delta)\,,$$
$$\mathbb{P}\big\{X_n \ne (s_{J(\theta^n\omega)}, \theta^n\omega) | X_0 \in S \times \{\omega\}\big\} \ge \varepsilon^\ell c_\delta(1 - c_\delta)\,,$$

provided that $0 \le \varepsilon < \varepsilon_\delta$. Consequently, for any two independent copies $\mathcal{X}, \mathcal{Y}$ of $\mathcal{X}^\varepsilon$ with $0 \le \varepsilon < \varepsilon_\delta$, and with $b_\delta := 2(c_\delta^2 + 1/c_\delta) > 2 + c_\delta$,

$$\begin{aligned}\mathbb{P}\big\{X_n = Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} &\ge \big(1 - \varepsilon^\ell(c_\delta^2 + 1/c_\delta)\big)^2 \\ &\ge 1 - \varepsilon^\ell b_\delta\,, \\ \mathbb{P}\big\{X_n \ne Y_n | X_0, Y_0 \in S \times \{\omega\}\big\} &\ge 2\big(1 - \varepsilon^\ell(c_\delta^2 + 1/c_\delta)\big)\varepsilon^\ell c_\delta(1 - c_\delta) \ge \varepsilon^\ell c_\delta\,,\end{aligned}$$

whenever $n \in D_\delta(\omega)$. In this case, all assertions of the theorem are correct with any $0 < a < 1$: Simply let $E_\delta(\omega)$ be a subset of $D_\delta(\omega)$ with density $a\mu(C_\delta) > a - a\delta > a - \delta$ and take $F_\delta(\omega) = D_\delta(\omega)\setminus E_\delta(\omega)$, with density $(1-a)\mu(C_\delta) > 1 - a - (1-a)\delta > 1 - a - \delta$. □

Remark 5.4. As the above proof shows, one may stipulate $b_\delta = c_\delta$ in Theorem C, except when $\mu\big(\Omega^{(i-1)} \setminus \Omega^{(i)}\big) = 1$ for some (necessarily unique) $1 \le i \le m$.

For a simple corollary, recall the first-order probability dissipation $d(\omega)$ from Sect. 4.

Corollary 5.5. *Let $\{\mathcal{X}^\varepsilon\}$ be a C^m-Markov perturbation ($m \ge 1$) of a synchronized DRN $\mathcal{X}^0$. If $\mu(\{\omega \in \Omega : d(\theta^{-1}\omega) = 0\}) < 1$ then the conclusions of Theorem C hold with $\ell = 1$ and $a = \mu\big(\Omega^{(1)}\big)$ if $\mu\big(\Omega^{(1)}\big) > 0$, or any $0 < a < 1$ if $\mu\big(\Omega^{(1)}\big) = 1$.*

Proof. By Lemma 4.4(i), $q^{(1)} \neq 0$ unless $d(\theta^{-1}\omega) = 0$. Thus $\mu\big(\Omega^{(1)}\big) < 1$, and the same arguments as in the proof of Theorem C show that $a = \mu\big(\Omega^{(1)}\big)$, provided that $\mu\big(\Omega^{(1)}\big) > 0$, and otherwise $0 < a < 1$ is arbitrary. □

6. An Example

This short final section illustrates some of the concepts and results of this work in the context of a concrete random network. Throughout, fix $k \ge 2$ and recall from Sect. 2 that $\mathcal{M}^+_{1,\text{det}}$ denotes the set of all deterministic stochastic $k \times k$-matrices; for convenience, henceforth write $\mathcal{M}^+_{1,\text{det}}$ as $\mathcal{M}$. Every $A \in \mathcal{M}$ corresponds to a unique map T_A of $S = \{s_1, \ldots, s_k\}$ into itself in a natural way, via

$$T_A(s_j) = s_i \quad \text{if and only if} \quad A_{i,j} = 1 \,. \tag{6.1}$$

Note that $\#T_A S = \operatorname{rank} A$. For every $\ell \in \mathbb{N}$ consider

$$\mathcal{M}_{[\ell]} := \{A \in \mathcal{M} : \operatorname{rank} A = \ell\} \,.$$

Obviously, $\mathcal{M}_{[\ell]} = \varnothing$ whenever $\ell > k$, and $\mathcal{M}$ is the disjoint union of $\mathcal{M}_{[1]}, \ldots, \mathcal{M}_{[k]}$. Moreover, $\#\mathcal{M} = k^k$ whereas

$$\#\mathcal{M}_{[\ell]} = \binom{k}{\ell} \sum\nolimits_{j=1}^{\ell} (-1)^{\ell-j} j^k \binom{\ell}{j} \quad \forall \ell \in \mathbb{N} \,;$$

in particular, $\#\mathcal{M}_{[1]} = k$ and $\#\mathcal{M}_{[k]} = k!$; see also Figure 3. Let ν be the uniform distribution on $\mathcal{M}$, that is, let $\nu(\{A\}) = 1/k^k$ for every $A \in \mathcal{M}$. (The reader will notice that all subsequent observations remain virtually unchanged as long as, more generally, $\nu(\{A\}) > 0$ for every $A \in \mathcal{M}$.) With these ingredients, consider the metric dynamical system $\Theta = (\Omega, \mathcal{F}, \mu, \theta)$, where the probability space is

$$(\Omega, \mathcal{F}, \mu) = \bigotimes\nolimits_{z \in \mathbb{Z}} (\mathcal{M}, 2^{\mathcal{M}}, \nu) \,,$$

and the map θ is the left shift

$$\theta\big((A_z)_{z\in\mathbb{Z}}\big) = (A_{z+1})_{z\in\mathbb{Z}} \quad \forall (A_z)_{z\in\mathbb{Z}} \in \Omega \,.$$

The dynamical system Θ, an example of a *Bernoulli shift*, is invertible and ergodic [14]. In probability theory parlance, Θ provides the canonical model of a (bi-infinite) sequence of i.i.d random variables uniformly distributed on $\mathcal{M}$. Note that every $\omega \in \Omega$ has the form $(A_z)_{z\in\mathbb{Z}}$, with $A_z \in \mathcal{M}$ for all z, and is often written as such in what follows.

To define a DRN $\mathcal{X}^0$ over Θ, let

$$T_{\mathcal{X}^0}(0, \omega) = \text{id}_S \,, \quad T_{\mathcal{X}^0}(n, \omega) = T_{A_{n-1}} \circ \cdots \circ T_{A_0} \quad \forall n \in \mathbb{N}, \omega \in \Omega \,, \tag{6.2}$$

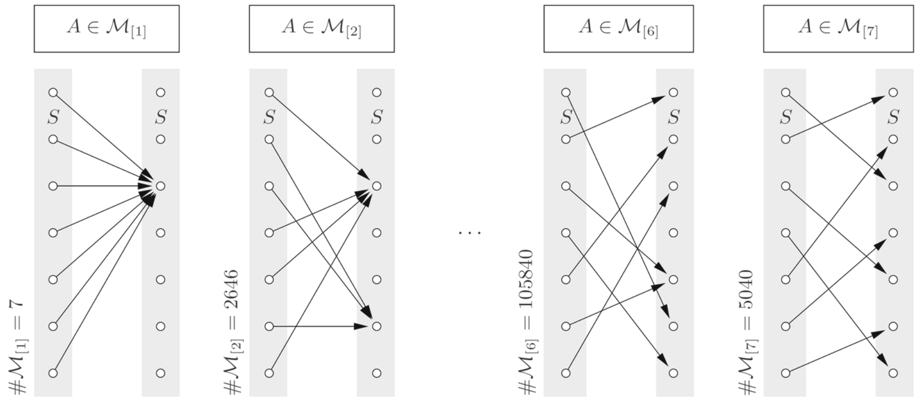

Fig. 3. Illustrating typical maps $T_A : S \to S$ associated with $A \in \mathcal{M}$ via (6.1), for $k = 7$ where $\#\mathcal{M} = 823543$

the transition cocycle of which simply is

$$P^0(n, \omega) = \begin{cases} I_k & \text{if } n = 0\,, \\ A_{n-1} \cdots A_0 & \text{if } n \geq 1\,. \end{cases}$$

Given $\omega \in \Omega$, notice that the sequence $\big(\text{rank}\, P^0(n, \omega)\big)_{n \in \mathbb{N}_0}$ in $\mathbb{K}$ is non-increasing and hence convergent. In particular, if $A_m \in \mathcal{M}_{[1]}$ for some $m \in \mathbb{N}_0$ then rank $P^0(n, \omega) = 1$ for all $n > m$. Since $\nu(\mathcal{M}_{[1]}) > 0$, this occurs for μ-a.e. $\omega \in \Omega$, and the function $N_0^+ : \Omega \to \mathbb{N}$ given by

$$N_0^+(\omega) = \begin{cases} \min\{n \in \mathbb{N} : \text{rank}\, P^0(n, \omega) = 1\} & \text{if } \lim_{n\to\infty} \text{rank}\, P^0(n, \omega) = 1\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

is well-defined. Plainly, rank $P^0(n, \omega) = 1$ for μ-a.e. $\omega \in \Omega$ and all $n \geq N_0^+(\omega)$, and hence N_0^+ is a forward synchronization time of $\mathcal{X}^0$. In fact, $N_0^+ \leq N^+$ for every forward synchronization time N^+ of $\mathcal{X}^0$; in particular, $\mathcal{X}^0$ is synchronized. Notice that, on the one hand, N_0^+ is unbounded, since for every $n \in \mathbb{N}$,

$$\mu(\{N_0^+ \geq n\}) \geq \mu\big(\{\omega : A_0, \ldots, A_{n-2} \in \mathcal{M}_{[k]}\}\big) = \nu(\mathcal{M}_{[k]})^{n-1} > 0\,.$$

On the other hand,

$$\mu(\{N_0^+ \geq n\}) \leq \mu\big(\{\omega : A_0, \ldots, A_{n-2} \notin \mathcal{M}_{[1]}\}\big) = \big(1 - \nu(\mathcal{M}_{[1]})\big)^{n-1}\,,$$

and consequently $\int_\Omega N_0^+ \,\mathrm{d}\mu \leq \sum_{n=1}^\infty \big(1 - \nu(\mathcal{M}_{[1]})\big)^{n-1} = 1/\nu(\mathcal{M}_{[1]}) < \infty$.

In a completely analogous manner, $\big(\text{rank}\, P^0(n, \theta^{-n}\omega)\big)_{n \in \mathbb{N}_0}$ is non-increasing for every $\omega \in \Omega$, and $N_0^- : \Omega \to \mathbb{N}$ given by

$$N_0^-(\omega) = \begin{cases} \min\{n \in \mathbb{N} : \text{rank}\, P^0(n, \theta^{-n}\omega) = 1\} & \text{if } \lim_{n\to\infty} \text{rank}\, P^0(n, \theta^{-n}\omega) = 1\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

is a pull-back synchronization time of $\mathcal{X}^0$, with $N_0^- \leq N^-$ for every pull-back synchronization time N^- of $\mathcal{X}^0$. It is readily seen that for every $n \in \mathbb{N}$ the sets $\{N_0^- \geq n\}$ and

$\theta^{n-1}\{N_0^+ \geq n\}$ differ only by a μ-nullset, and so $\mu(\{N_0^- = n\}) = \mu(\{N_0^+ = n\})$, even though clearly $\mu(\{N_0^- \neq N_0^+\}) \geq 2\nu(\mathcal{M}_{[1]})\big(1 - \nu(\mathcal{M}_{[1]})\big) > 0$. As far as they pertain to the dynamics of $\mathcal{X}^0$, the above observations yield

Proposition 6.1. *The DRN $\mathcal{X}^0$ defined in (6.2) is synchronized, with synchronization index $J : \Omega \to \mathbb{K}$ given by $e_J(\omega) = \lim_{n\to\infty} A_{-1}\cdots A_{-n}e_1$ for μ-a.e. $\omega = (A_z)_{z\in\mathbb{Z}} \in \Omega$. An $\mathcal{F}$-measurable function $N : \Omega \to \mathbb{N}$ is a forward (or pull-back) synchronization time of $\mathcal{X}^0$ if and only if $N \geq N_0^+$ (or $N \geq N_0^-$) μ-a.e. on Ω; in particular, $\mathcal{X}^0$ is not uniformly synchronized.*

To construct a Markov perturbation of $\mathcal{X}^0$, recall from Sect. 2 that $\mathcal{M}_0$ denotes the linear space of all real zero-column-sum $k \times k$-matrices, and consider any function $f : \mathcal{M} \to \mathcal{M}_0$. Notice that $A + \varepsilon f(A) \in \mathcal{M}_1^+$ for all sufficiently small $\varepsilon \geq 0$ if and only if

$$f(A)_{i,j}(1 - 2A_{i,j}) \geq 0 \quad \forall i, j \in \mathbb{K}\,. \tag{6.3}$$

Motivated by (6.3), fix any $f : \mathcal{M} \to \mathcal{M}_0$ with the property that for each $A \in \mathcal{M}$ either $f(A) = 0$, or else

$$f(A)_{i,j}(1 - 2A_{i,j}) > 0 \quad \forall i, j \in \mathbb{K}\,; \tag{6.4}$$

assume w.l.o.g. that $|f(A)| \leq 1$ as well. For every $\varepsilon \geq 0$, $n \in \mathbb{N}$, and $\omega \in \Omega$, letting

$$P_{\mathcal{X}^\varepsilon}(n, \omega) = \big(A_{n-1} + \min\{\varepsilon, 1\}f(A_{n-1})\big)\cdots\big(A_0 + \min\{\varepsilon, 1\}f(A_0)\big) \tag{6.5}$$

then yields an MRN $\mathcal{X}^\varepsilon$, in fact a C^∞-Markov perturbation of $\mathcal{X}^0$. Notice that by the strict inequality in (6.4), each state of X_n in $S \times \{\theta^n\omega\}$ has a positive probability of leading to any state of X_{n+1} in $S \times \{\theta^{n+1}\omega\}$, unless $f(A_n) = 0$. Regardless of the specific choice of f, Theorems A and B apply. Moreover, deduce from $P^{(1)}_{\mathcal{X}^\varepsilon}(1, \omega) = f(A_0)$ and (4.4) that

$$P^{(1)}_{\mathcal{X}^\varepsilon}(n, \omega) = \sum\nolimits_{\ell=0}^{n-1} A_{n-1}\cdots A_{\ell+1} f(A_\ell) A_{\ell-1}\cdots A_0\,,$$

and hence the quantities d and $q^{(1)}$ introduced in Sect. 4 now read

$$\begin{aligned} d(\omega) &= f(A_0)e_{J(\omega)}\,, \\ q^{(1)}(\omega) &= \sum\nolimits_{\ell=0}^{N_0^-(\omega)-1} A_{-1}\cdots A_{\ell+1-N_0^-(\omega)} f(A_{\ell-N_0^-(\omega)}) e_{J(\theta^{\ell-N_0^-(\omega)}\omega)}\,. \end{aligned} \tag{6.6}$$

Also, the set $S_\bullet \subset S$ is $S_\bullet(\omega) = \{s_j : A_0 e_j = e_{J(\theta\omega)}\}$, and hence $S_\bullet(\omega) \neq S$ unless $A_0 \in \mathcal{M}_{[1]}$. Since either all components of $d(\omega)$ are zero or else none is, by virtue of (6.4),

$$\Omega_\bullet = \{\omega : f(A_0) = 0 \text{ or } A_1 \in \mathcal{M}_{[1]}\}\,.$$

Finally, by utilizing (6.6) the set of first-order degeneracy $\Omega^{(1)} = \{q^{(1)} = 0\}$ considered in Sect. 5 can be described explicitly also.

Proposition 6.2. *Let $\mathcal{X}^\varepsilon$ be the MRN defined in (6.5). Then, for all sufficiently small $\varepsilon > 0$,*

$$\Omega^{(1)} = \{\omega = (A_z)_{z\in\mathbb{Z}} : f(A_{-1}) = \ldots = f(A_{-N_0^-(\omega)}) = 0\}\,,$$

up to a μ-nullset.

Notice that the value of $\mu\big(\Omega^{(1)}\big) \le \nu(\{f = 0\})$ is completely determined by the set $\{f = 0\} \subset \mathcal{M}$. For instance, $\mu\big(\Omega^{(1)}\big) = 1$ if and only if $\{f = 0\} = \mathcal{M}$, and $\mu\big(\Omega^{(1)}\big) = \nu(\{f = 0\})$ if $\{f = 0\} \subset \mathcal{M}_{[1]}$. Apart from trivial situations like these, computing the exact value of $\mu(\Omega^{(1)})$ may be a challenge. (In probability theory parlance, $\mu\big(\Omega^{(1)}\big)$ equals the probability that for a sequence $(A_1, A_2, \ldots)$ of matrices, chosen independently and uniformly from $\mathcal{M}$, the product $A_n \cdots A_1$ attains rank 1 *before* $f(A_n) \neq 0$ for the first time.) Regardless of the exact value, however, Theorem C applies with $\ell = 1$ whenever $\{f = 0\} \neq \mathcal{M}$. Provided that $f : \mathcal{M} \to \mathcal{M}_0$ is not identically zero (and satisfies (6.4) unless $f(A) = 0$), therefore, the MRN $\mathcal{X}^{\varepsilon}$ defined in (6.5) for all sufficiently small $\varepsilon > 0$ does exhibit the intermittency between high-probability synchronization and low-probability desynchronization established in that theorem.

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