



The mathematics of Benford's law: a primer

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Abstract

This article provides a concise overview of the main mathematical theory of Benford's law in a form accessible to scientists and students who have had first courses in calculus and probability. In particular, one of the main objectives here is to aid researchers who are interested in applying Benford's law, and need to understand general principles clarifying when to expect the appearance of Benford's law in real-life data and when not to expect it. A second main target audience is students of statistics or mathematics, at all levels, who are curious about the mathematics underlying this surprising and robust phenomenon, and may wish to delve more deeply into the subject. This survey of the fundamental principles behind Benford's law includes many basic examples and theorems, but does not include the proofs or the most general statements of the theorems; rather it provides precise references where both may be found.

Keywords Benford's law · Significant · Scale invariance · Base invariance · Random probability measure

1 Introduction

Applications of the well-known statistical phenomenon called *Benford's law*, or *first-digit law*, have been increasing dramatically in recent years. The online Benford database Berger et al. (2009), for example, shows over 800 new entries in the past decade alone. At the *Cross-domain Conference on Benford's Law Applications* hosted by the Joint Research Centre of the European Commission in Stresa, Italy in July 2019, organizers and participants both expressed a need for a readily available and relatively non-technical summary of the mathematics

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underlying Benford's law. This article is an attempt to satisfy that request. As such, this overview of the mathematics of Benford's Law is formulated without relying on more advanced concepts from such mathematical fields as measure theory and complex analysis.

The topic of Benford's law has a rich and fascinating history. First recorded in the 19th century (Newcomb 1881), it is now experiencing a wide variety of applications including detection of tax and voting fraud, analysis of digital images, and identification of anomalies in medical, physical, and macroeconomic data, among others. The interested reader is referred to Berger and Hill (2015); Miller (2015); Nigrini (2012) for more extensive details on the history and applications of Benford's law.

It is our hope that the present Benford primer will be useful for two groups of readers in particular: First, researchers who are interested in applying Benford's law, and need to understand general principles clarifying when to expect the appearance of Benford's law in real-life data, and when not to expect it; and second, science students at both the undergraduate and graduate levels who are curious about the mathematical basis for this surprising phenomenon, and may wish to delve more deeply into the subject and perhaps even try their hands at solving some of the open problems.

This survey includes special cases of most of the main Benford theorems, and many concrete examples, but does not include proofs or the most general statements of the theorems, most of which may be found as indicated in Berger and Hill (2015). The structure of the article is as follows: Sect. 2 contains the notation and definitions; Sect. 3 the basic properties that characterize Benford behavior; Sect. 4 the Benford properties of sequences of constants; Sect. 5 the Benford properties of sequences of random variables; and Sect. 6 a brief discussion of four common errors.

2 Basic notation and definitions

In this survey, the emphasis is on *decimal* representations of numbers, the classical setting of Benford's law, so here and throughout $\log t$ means $\log_{10} t$, and all digits are *decimal* digits. For other bases such as binary or hexadecimal, analogous results hold with very little change, simply by replacing \log with \log_b for the appropriate base b ; the interested reader is referred to (Berger and Hill 2015, p. 9) for details.

Here and throughout, $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the positive integers (or natural numbers), $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ the integers, $\mathbb{R} = (-\infty, \infty)$ the real numbers, and $\mathbb{R}^+ = (0, \infty)$ the positive real numbers. For real numbers a and b , $[a, b)$ denotes the set (in fact, half-open interval) of all $x \in \mathbb{R}$ with $a \leq x < b$; similarly for $(a, b]$, (a, b) , $[a, b]$. Every real number x can be expressed uniquely as $x = [x] + \langle x \rangle$, where $[x]$ and $\langle x \rangle$ denote the *integer part* and the *fractional part* of x , respectively. Formally, $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$ and $\langle x \rangle = x - [x]$. For example, $[2] = 2$ and $\langle 2 \rangle = 0$, whereas $[10\pi] = [31.4\dots] = 31$ and $\langle 10\pi \rangle = 0.415\dots$

The basic notion underlying Benford's law concerns the *leading significant digits* and, more generally, the *significantand* of a number (also sometimes referred to as the *mantissa* in scientific notation).

Definition 1 For $x \in \mathbb{R}^+$, the (*decimal*) *significantand* of x , denoted $S(x)$, is given by $S(x) = t$, where t is the unique number in $[1, 10)$ with $x = 10^k t$ for some (necessarily unique) $k \in \mathbb{Z}$. For negative x , $S(x) = S(-x)$, and for convenience, $S(0) = 0$.

Example 2 $S(2019) = 2.019 = S(0.02019) = S(-20.19)$.

Definition 3 The *first (decimal) significant digit* of $x \in \mathbb{R}$, denoted $D_1(x)$, is the first (left-most) digit of $S(x)$, where by convention the terminating decimal representation is used if $S(x)$ has two decimal representations. Similarly, $D_2(x)$ denotes the second digit of $S(x)$, $D_3(x)$ the third digit of $S(x)$, and so on. (Note that $D_n(0) = 0$ for all $n \in \mathbb{N}$.)

Example 4 $D_1(2019) = D_1(0.02019) = D_1(-20.19) = 2$, $D_2(2019) = 0$, $D_3(2019) = 1$, $D_4(2019) = 9$, and $D_j(2019) = 0$ for all $j \geq 5$. Also, $D_n(2019) = D_n(2018.9999\dots)$ for all $n \in \mathbb{N}$.

As will be seen next, the formal notions of a Benford sequence of numbers and a Benford random variable are defined via the significantands, or equivalently, via the significant digits of the sequence and the random variable.

An infinite sequence of real numbers (x_1, x_2, x_3, \dots) is denoted by (x_n) ; e.g., $(2^n) = (2, 2^2, 2^3, \dots) = (2, 4, 8, \dots)$. In the next definition, $\#A$ denotes the number of elements of the set A ; e.g., $\#\{2, 0, 1, 9\} = 4$.

Definition 5 A sequence of real numbers (x_n) is a *Benford sequence*, or *Benford* for short, if for every $t \in [1, 10)$, the limiting proportion of x_n 's with significantand less than or equal to t is exactly $\log t$, i.e., if

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : S(x_n) \leq t\}}{N} = \log t \quad \text{for all } t \in [1, 10).$$

Example 6 (1) The sequence of positive integers $(n) = (1, 2, 3, \dots)$ is not Benford, since, for example, more than half the entries less than $2 \cdot 10^m$ have first digit 1 for every positive integer m , so the limiting proportion of entries with significantand less than or equal to 2, if it exists at all, cannot be $\log 2 < 0.5$. Similarly, the sequence of prime numbers $(2, 3, 5, 7, 11, \dots)$ is not Benford, but the demonstration of this fact is deeper; see (Berger and Hill 2015, Example 4.17(v)).

(2) As will be seen in Example 15 below, the sequences (2^n) and (3^n) of powers of 2 and 3 are Benford. Many other classical sequences including the Fibonacci sequence $(1, 1, 2, 3, 5, \dots)$ and the sequence of factorials $(n!) = (1, 2, 6, 24, 120, \dots)$ are also Benford.

An equivalent description of a Benford sequence in terms of the limiting proportions of values of its significant digits is as follows.

Proposition 7 A sequence (x_n) of real numbers is Benford if and only if

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : D_1(x_n) = d_1, D_2(x_n) = d_2, \dots, D_m(x_n) = d_m\}}{N} = \log \left(1 + \frac{1}{10^{m-1}d_1 + 10^{m-2}d_2 + \dots + d_m} \right),$$

for all $m \in \mathbb{N}$, all $d_1 \in \{1, 2, \dots, 9\}$, and all $d_j \in \{0, 1, \dots, 9\}$, $j \geq 2$.

Example 8 Proposition 7 with $m = 1$ yields the well-known *first-digit law*: For every Benford sequence of real numbers (x_n) ,

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : D_1(x_n) = d\}}{N} = \log \left(1 + \frac{1}{d} \right) \quad \text{for all } d \in \{1, 2, \dots, 9\}.$$

The notion of a Benford random variable (or dataset) is essentially the same as that of a Benford sequence, with the limiting proportion of entries replaced by the *probability* of the random values.

Definition 9 A (real-valued) random variable X is *Benford* if

$$P(S(X) \leq t) = \log t \quad \text{for all } t \in [1, 10).$$

Recall that a random variable U is said to be *uniformly distributed* on $[0, 1]$ if $P(U \leq s) = s$ for all $s \in [0, 1]$.

Example 10 Let U be uniformly distributed on $[0, 1]$.

- (1) U is not Benford, since as is easy to check, $P(S(U) \leq 2) = \frac{1}{9} < \log 2$.
- (2) $X = 10^U$ is Benford, since $S(X) = X$, and $P(S(X) \leq t) = P(X \leq t) = P(10^U \leq t) = P(U \leq \log t) = \log t$ for all $t \in [1, 10)$. In fact, this construction provides an excellent way of generating random data that follows Benford's law on a digital computer: Use any standard program to generate U , and then raise 10 to that power.

The analogous definition of a Benford random variable in terms of significant digits follows similarly.

Proposition 11 A random variable X is Benford if and only if

$$P(D_1(X) = d_1, D_2(X) = d_2, \dots, D_m(X) = d_m) = \log \left(1 + \frac{1}{10^{m-1}d_1 + 10^{m-2}d_2 + \dots + d_m} \right),$$

for all $m \in \mathbb{N}$, all $d_1 \in \{1, 2, \dots, 9\}$, and all $d_j \in \{0, 1, \dots, 9\}$, $j \geq 2$.

Example 12 If X is a Benford random variable, then the probability that X has the same first three digits as $\pi = 3.1415\dots$ is

$$\begin{aligned} P(D_1(X) = 3, D_2(X) = 1, D_3(X) = 4) &= \log\left(1 + \frac{1}{10^2 \cdot 3 + 10 \cdot 1 + 4}\right) \\ &= \log\frac{315}{314} \approx 0.00138. \end{aligned}$$

None of the classical random variables are Benford exactly, although some are close for certain values of their parameters. For example, no uniform, exponential, normal, or Pareto random variable is Benford exactly, but Pareto and log normal random variables, among others, can be arbitrarily close to being Benford depending on the values of their parameters.

3 What properties characterize Benford sequences and random variables?

The purpose of this section is to exhibit several fundamental and useful results concerning Benford sequences and random variables. These include three basic properties of a sequence of constants or a random variable that are equivalent to it being Benford:

- (1) the fractional parts of its decimal logarithm are uniformly distributed between 0 and 1;
- (2) the distribution of its significant digits is invariant under changes of scale; and
- (3) the distribution of its significant digits is continuous and invariant under changes of base.

Analogous definitions and results also hold for *Benford functions*, for which the interested reader is referred to (Berger and Hill 2015, Sect. 3.2).

An additional feature demonstrating the robustness of Benford's law is that if a Benford random variable is multiplied by any independent positive random variable, then the product is Benford as well.

Recall that a sequence of real numbers $(x_n) = (x_1, x_2, x_3, \dots)$ is *uniformly distributed modulo one* (or *mod 1*, for short) if

$$\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : \langle x_n \rangle \leq s\}}{N} = s \quad \text{for all } s \in [0, 1],$$

e.g., in the limit, exactly half of the fractional parts $\langle x_n \rangle$ are less than or equal to $\frac{1}{2}$, and exactly one third are less than or equal to $\frac{1}{3}$. The next lemma is a classical equidistribution theorem of Weyl, and, as will be seen, is a powerful tool in Benford theory.

Lemma 13 *The sequence $(na) = (a, 2a, 3a, \dots)$ is uniformly distributed mod 1 if and only if a is irrational.*

Proof See (Berger and Hill 2015, Proposition 4.6). □

The application of Lemma 13 to the theory of Benford's law is evident from the following basic characterization of Benford sequences. (Here and throughout let $\log 0 = 0$ for convenience.)

Theorem 14 A sequence of real numbers (x_n) is Benford if and only if the sequence $(\log |x_n|) = (\log |x_1|, \log |x_2|, \log |x_3|, \dots)$ is uniformly distributed mod 1.

Proof See (Berger and Hill 2015, Theorem 4.2). □

Example 15 (1) The sequence (2^n) of powers of 2 is Benford. This follows by Theorem 14 and Lemma 13 since $(\log 2^n) = (n \log 2)$ and since $\log 2$ is irrational. Similarly, the sequences (3^n) and (5^n) of powers of 3 and 5, respectively, are Benford.

(2) The sequence (10^n) is not Benford, nor is $(10^{n/2}) = (\sqrt{10}, 10, 10\sqrt{10}, \dots)$, since $\langle \log 10^{n/2} \rangle = \langle \frac{n}{2} \rangle = 0$ or $\frac{1}{2}$ for every n , so $(\log 10^{n/2})$ is not uniformly distributed mod 1.

The following characterization of Benford random variables is a direct analogue of Theorem 14.

Theorem 16 A random variable X is Benford if and only if the random variable $\langle \log |X| \rangle$ is uniformly distributed on $[0, 1]$.

Proof See (Berger and Hill 2015, Theorem 4.2). □

The next proposition shows that if a sequence of numbers or a random variable are Benford, then so are the positive multiples of the sequence or random variable, as are their powers and reciprocals.

Proposition 17 If the sequence of numbers (x_n) is Benford, and if the random variable X is Benford, then for every $a > 0$ and $0 \neq k \in \mathbb{Z}$, the sequence (ax_n^k) and the random variable (aX^k) are also Benford.

Proof Special case of (Berger and Hill 2015, Theorem 4.4). □

Example 18 (1) Since (2^n) is Benford, the sequences $(4^n) = (4, 16, 64, \dots)$, $(2^{-n}) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$, and $(2^n \pi) = (2\pi, 4\pi, 8\pi, \dots)$ are also Benford.

(2) Since $X = 10^U$ is Benford, so are $X^2 = 100^U$, $1/X = 10^{-U}$, and $\pi X = \pi 10^U$.

The next theorem says that if a Benford random variable is multiplied by any positive constant, e.g., as a result of changing units of measurement, then the significant digit probabilities will not change. In fact Benford random variables are the only random variables with this property. Recall that two random variables X and Y are *identically distributed* if $P(X \leq t) = P(Y \leq t)$ for all $t \in \mathbb{R}$.

Definition 19 A random variable X has *scale-invariant significant digits* if $S(X)$ and $S(aX)$ are identically distributed for all $a \in \mathbb{R}^+$.

Example 20 Let U be uniformly distributed on $[0, 1]$.

- (1) U does not have scale-invariant digits since, for example, $P(S(U) \leq 2) = \frac{1}{9}$ but $P(S(2U) \leq 2) = \frac{5}{9}$.
- (2) As is easy to check directly, or follows immediately from the next theorem and Example 10 above, the random variable $X = 10^U$ has scale-invariant significant digits.

Theorem 21 A random variable X with $P(X = 0) = 0$ is Benford if and only if it has scale-invariant significant digits.

Proof See (Berger and Hill 2015, Theorem 5.3). □

Example 22 By Theorem 21 and Example 10 above, if U is uniformly distributed on $[0, 1]$, then for every $a > 0$ the random variable aU is not Benford, whereas the random variable $a10^U$ is Benford.

In fact, a much weaker form of scale-invariance characterizes Benford's law completely, namely, scale-invariance of any single first digit.

Theorem 23 A random variable X with $P(X = 0) = 0$ is Benford if and only if for some $d \in \{1, 2, \dots, 9\}$,

$$P(D_1(aX) = d) = P(D_1(X) = d) \quad \text{for all } a \in \mathbb{R}^+.$$

Proof See (Berger and Hill 2015, Theorem 5.8). □

Example 24 If X is a positive random variable, and the probability that the first significant digit of aX equals 3 is the same for all $a \in \mathbb{R}^+$, then X is Benford.

A notion parallel to that of scale-invariance is the notion of *base-invariance*, one interpretation of which says that the distribution of the significant digits remains unchanged if the base is changed from 10 to, say, 100.

Definition 25 A random variable X has *base-invariant significant digits* if $S(X)$ and $S(X^n)$ are identically distributed for all $n \in \mathbb{N}$.

Example 26 Let U be uniformly distributed on $[0, 1]$.

- (1) A short calculation (e.g., see (Berger and Hill 2015, Example 5.11(3))) shows that U does not have base-invariant significant digits.
- (2) A random variable Y with $P(S(Y) = 1) = 1$ clearly has base-invariant significant digits, as does any Benford random variable, which follows by a short calculation; see (Berger and Hill 2015, Example 5.11(2)).

As seen in the last example, random variables whose significand equals 1 with probability one, and Benford random variables both have base-invariant significant digits. In fact, as the next theorem shows, averages of these two distributions are the only such random variables.

Theorem 27 A random variable Z with $P(Z = 0) = 0$ has base-invariant significant digits if and only if $Z = (1 - q)X + qY$ for some $q \in [0, 1]$, where X is Benford and $P(S(Y) = 1) = 1$.

Proof See (Berger and Hill 2015, Theorem 5.13). □

Theorem 28 If a random variable has scale-invariant significant digits then it has base-invariant significant digits.

Proof Follows immediately from Theorems 21 and 27. □

A consequence of Theorem 27 is that there are many base-invariant random variables that are *not* Benford, but as the next corollary shows, all continuous random variables that are base-invariant are also Benford. Recall that a random variable X is *continuous* if there exists a function $f_X : \mathbb{R} \rightarrow [0, \infty)$, the *density function* of X , such that

$$P(X \leq t) = \int_{-\infty}^t f_X(x) dx \quad \text{for all } t \in \mathbb{R}.$$

As the reader may notice, such a random variable X is often called *absolutely continuous* in advanced texts, whereas the term *continuous* refers to the (weaker) property that $P(X = t) = 0$ for all $t \in \mathbb{R}$. In keeping with the elementary nature of this article, random variables that have the latter property but not the former (such as, e.g., Cantor random variables (Berger and Hill 2015, Example 8.9)) are not considered here, and *continuous* means *absolutely continuous* throughout. Many of the most common and useful random variables are continuous, including uniform, normal, and exponential random variables. Every Benford random variable is continuous.

Corollary 29 A continuous random variable is Benford if and only if it has base-invariant significant digits.

The final theorem in this section illustrates one of the key “attracting” properties of Benford random variables, namely, if any random variable is multiplied by an independent Benford random variable, then the product is Benford.

Theorem 30 Let X, Y be independent random variables with $P(XY=0) = 0$. If either X or Y is Benford, then the product XY is also Benford.

Proof See (Berger and Hill 2015, Theorem 8.12). □

Corollary 31 Let X_1, X_2, \dots be independent positive random variables. If X_j is Benford for some $j \in \mathbb{N}$, then the product $X_1 X_2 \cdots X_m$ is Benford for all $m \geq j$.

4 What sequences of constants are Benford?

The goal of this section is to describe the Benford behavior of deterministic (that is, non-random) sequences. The sequences described below will typically be increasing (or decreasing) sequences of positive constants given by a rule that specifies the next

entry in the sequence as a function of the previous entry (or several previous entries, for example, as in the Fibonacci sequence). The most common examples are iterations of a single function, i.e., where the same function is applied over and over again. As will be seen here, three basic principles describe the Benford behavior of such sequences:

- (1) no polynomially increasing or decreasing sequence (or its reciprocals) is Benford;
- (2) almost every, but not every, exponentially increasing positive sequence is Benford, and if it is Benford for one starting point, then it is Benford for all starting points; and
- (3) every super-exponentially increasing or decreasing positive sequence is Benford for almost every, but not every, starting point.

To facilitate discussion of iterations of a function $f : \mathbb{R} \rightarrow \mathbb{R}$, the n th iterate of f is denoted by $f^{[n]}$, so $f^{[1]}(x) = f(x)$, $f^{[2]}(x) = f(f(x))$, $f^{[3]}(x) = f(f(f(x)))$, etc. Thus, $(f^{[n]}(x))$ denotes the infinite sequence of iterates of f starting at x , i.e.,

$$(f^{[n]}(x)) = (f(x), f(f(x)), f(f(f(x))), \dots).$$

The next example illustrates sequences with the three types of growth mentioned above.

Example 32

- (1) Let $f(x) = x + 1$. Then $(f^{[n]}(x)) = (x + 1, x + 2, x + 3, \dots)$, so $(f^{[n]}(1)) = (2, 3, 4, \dots)$, a polynomially (in fact, linearly) increasing sequence.
- (2) Let $g(x) = 2x$. Then $(g^{[n]}(x)) = (2x, 4x, 8x, \dots)$, so $(g^{[n]}(1)) = (2, 4, 8, \dots)$ and $(g^{[n]}(3)) = (6, 12, 24, \dots)$, both exponentially increasing sequences.
- (3) Let $h(x) = x^2$. Then $(h^{[n]}(x)) = (x^2, x^4, x^8, \dots)$. Then $(h^{[n]}(1)) = (1, 1, 1, \dots)$ is constant whereas $(h^{[n]}(2)) = (4, 16, 256, \dots)$ is a super-exponentially increasing sequence.

Recall from Example 6(1) that the sequence of positive integers (n) is not Benford. Thus by the scale-invariance characterization of Benford sequences in Theorem 21 above, no arithmetic sequence $(a, 2a, 3a, \dots)$ is Benford for any real number a either. In fact, no polynomially increasing sequence, or the decreasing sequence of its reciprocals, is Benford.

Proposition 33 The sequence $(an^b) = (a, a2^b, a3^b, \dots)$ is not Benford for any real numbers a and b .

Proof See (Berger and Hill 2015, Example 4.7(ii)). □

Example 34 The sequences $(n^2) = (1, 4, 9, \dots)$ and $(n^{-2}) = (1, \frac{1}{4}, \frac{1}{9}, \dots)$ are not Benford.

Recall again that the sequence (2^n) is Benford. This also follows as a special case from the next theorem, which deals with exponentially increasing sequences generated by iterations of linear functions. Recall that a real number a is a *rational power* of 10 if $a = 10^{m/k}$ for some $m, k \in \mathbb{Z}, k \neq 0$. For example, $\sqrt{10} = 10^{1/2}$ and $\sqrt[3]{100} = 10^{2/3}$ are rational powers of 10, but 2 and π are not. As is easy to check, if X is a continuous random variable, then $P(X \text{ is a rational power of } 10) = 0$.

Theorem 35 Let $f(x) = ax + b$ for some real numbers $a > 1$ and $b \geq 0$. Then for every $x > 0$ the sequence $(f^{[n]}(x))$ is Benford if and only if a is not a rational power of 10.

Proof See (Berger and Hill 2015, Theorem 6.13). □

Example 36

- (1) Let $f(x) = 2x$. Since 2 is not a rational power of 10, the sequence $(f^{[n]}(x)) = (2^n x)$ is Benford for every $x > 0$; in particular taking $x = 1$ shows that (2^n) is Benford. Similarly, letting $g(x) = 2x + 1$, the sequence $(g^{[n]}(x)) = (2x + 1, 4x + 3, 8x + 7, \dots)$ is also Benford for every $x > 0$.
- (2) Let $g(x) = \sqrt{10}x$. Since $\sqrt{10} = 10^{1/2}$ is a rational power of 10, the sequence $(g^{[n]}(x)) = (\sqrt{10}x, 10x, 10\sqrt{10}x, \dots)$ is not Benford for any x . In particular, if $x = 1$, the first significant digit of every entry in the sequence is either 1 or 3.

The Benford behavior of sequences generated by iterations of linear functions as shown in Theorem 35, such as (x_n) where $x_{n+1} = 2x_n + 1$ for all $n > 1$, has been extended to various wider settings. One such setting is *linear difference equations*, where the next entry in a sequence may depend linearly on several past entries, such as the Fibonacci sequence $(1, 1, 2, 3, 5, \dots)$ where $x_{n+1} = x_n + x_{n-1}$; see (Berger and Hill 2015, Sect. 7.5).

As seen in Theorem 35 above, for exponentially increasing sequences generated by iterations of linear functions, the resulting sequence is Benford or not Benford depending on the coefficient of the leading term, and if it is Benford (or not Benford) for one starting point $x > 0$, then it is Benford (not Benford, respectively) for *all* starting points $x > 0$. As will be seen in the next theorem, this is in contrast to the situation for *super-exponentially* increasing (or decreasing) functions, where the Benford property of the sequence $(f^{[n]}(x))$ does not depend on the coefficient of the leading term, but does depend on the starting point x .

Theorem 37 Let f be any non-linear polynomial with $f(x) > x$ for some real number a and all $x > a$. Then $(f^{[n]}(X))$ is a Benford sequence with probability one for every continuous random variable X with $P(X > a) = 1$, but there are infinitely many $x > a$ for which $(f^{[n]}(x))$ is not Benford.

Proof See (Berger and Hill 2015, Theorem 6.23). \square

Thus super-exponentially increasing sequences are Benford for *almost all starting points* in the sense that if the starting point is selected at random according to any continuous distribution on $[a, \infty)$, then the resulting sequence is Benford with probability one.

Example 38 (1) Let $f(x) = x^2 + 1$. Note that $f(x) > x$ for all x , so in Theorem 37 the number a is arbitrary (or, more formally, one may take $a = -\infty$). Thus there are infinitely many x for which $(f^{[n]}(x))$ is not Benford, but $(f^{[n]}(X))$ is Benford with probability one if X is continuous. However, in this example it is not easy to determine exactly which starting points will yield Benford sequences. For instance, it is unknown whether or not the sequence starting at 1, i.e., $(f^{[n]}(1)) = (2, 5, 26, \dots)$, is Benford; see (Berger and Hill 2015, Example 6.25).

(2) Let $g(x) = x^2$. Here Theorem 37 applies with $a = 1$. Hence there are infinitely many $x > 1$ so that $(g^{[n]}(x)) = (x^2, x^4, x^8, \dots)$ is *not* Benford (e.g., $x = 10, 100, 1000, \dots$). Since $g^{[n]}(1/x) = 1/g^{[n]}(x) > 0$ for all $n \in \mathbb{N}$ and $x \neq 0$, it follows with Proposition 17 that if the starting point is selected at random via any continuous random variable X , then $(g^{[n]}(X)) = (X^2, X^4, X^8, \dots)$ is Benford with probability one.

The results for iterations of functions above deal exclusively with repeated application of the *same* function. As another example of the remarkable robustness of Benford's law, Benford sequences may also arise from the iterated application of *different* functions. The next proposition, which follows easily from (Berger and Hill 2015, Proposition 4.6(1)) and Theorem 14 above, provides an example of this behavior.

Proposition 39 Let $f_1(x) = a_1x + b_1$ and $f_2(x) = a_2x + b_2$ for some real numbers $a_1, a_2 > 1$ and $b_1, b_2 \geq 0$. Letting $g_n(x) = f_1(x)$ if n is odd, and $= f_2(x)$ if n is even, then for every $x > 0$ the sequence $(g^{[n]}(x)) = (g_1(x), g_2(g_1(x)), \dots)$ is Benford if and only if a_1a_2 is not a rational power of 10.

Example 40 Alternating multiplication by 2 and by 3 yields a Benford sequence for all starting points $x > 0$. In particular starting at $x = 1$, the sequence $(2, 6, 12, 36, 72, \dots)$ is Benford.

In the last example, since iterations of each of the functions $f_1(x) = 2x$ and $f_2(x) = 3x$ both lead to Benford sequences, it is perhaps not surprising that alternating applications of them also leads to a Benford sequence for every starting point $x > 0$. Similarly, even if the selection of applying f_1 or f_2 is done *at random* by flipping a fair coin at each step, the same conclusion holds (see Example 49 below). More surprisingly perhaps, even in situations where f_1 on its own would not generate any Benford sequences at all, and is applied more than half the time, the resulting sequence $(g^{[n]}(x))$ may still be Benford for most $x > 0$.

Example 41 Let $f_1(x) = \sqrt{x}$ and $f_2(x) = x^3$. Then $(f_1^{[n]}(x))$ is not a Benford sequence for any $x > 0$, since $(f_1^{[n]}(x)) = (\sqrt{x}, \sqrt[4]{x}, \sqrt[8]{x}, \dots)$ converges to 1 as $n \rightarrow \infty$. By Theorem 37 and Proposition 17, on the other hand, $(f_2^{[n]}(x))$ is a Benford sequence for almost all $x > 0$. As shown in (Berger and Hill 2015, Example 8.48), however, if the functions f_1 and f_2 are applied randomly and independently at each step, with f_1 applied no more than 61.3 percent of the time, then almost all of the sequences generated are Benford.

5 What sequences of random variables are Benford?

The goal of this section is to identify several of the key Benford limiting properties of sequences of random variables. These include the three basic facts that

- (1) powers of every continuous random variable converge to Benford's law;
- (2) products of random samples from every continuous distribution converge to Benford's law; and
- (3) if random samples are taken from random distributions that are chosen in an unbiased way, then the combined sample converges to Benford's law.

Here and throughout, *i.i.d.* stands for *independent and identically distributed*; by definition, a *random sample* is a finite sequence X_1, X_2, \dots, X_n of i.i.d. random variables.

Definition 42 An infinite sequence of random variables (X_1, X_2, X_3, \dots) converges in distribution to Benford's law if

$$\lim_{n \rightarrow \infty} P(S(X_n) \leq t) = \log t \quad \text{for all } t \in [1, 10),$$

and is Benford with probability one if

$$P((X_1, X_2, X_3, \dots) \text{ is a Benford sequence}) = 1.$$

In general, neither form of convergence implies the other, as the next example shows.

Example 43

- (1) Let X be a Benford random variable, and for each $n \in \mathbb{N}$, let $X_n = X$. Then the sequence $(X_n) = (X, X, X, \dots)$ converges to Benford's law in distribution, since $P(S(X_n) \leq t) = \log t$ for all n and all $t \in [1, 10)$. But (X_n) is never a Benford sequence, since no constant sequence is Benford.
- (2) Let X be a random variable that is identically 2, and let $X_n = X^n$ for all $n \in \mathbb{N}$. Then $(X_n) = (2^n)$ is Benford with probability one since (2^n) is a Benford sequence. But for every $n \in \mathbb{N}$, $X_n = 2^n$ is constant, which implies, for example, that $P(D_1(X_n) = 1) = 0$ or 1, and hence does not converge to the

Benford probability $\log 2$. Thus the sequence (X_n) does not converge in distribution to Benford's law.

- (3) If X_1, X_2, \dots are i.i.d. random variables, then it is easy to see that the sequence (X_n) converges in distribution to Benford's law if and only if it is Benford with probability one.

The next two theorems identify classical stochastic settings in which sequential products of random variables converge in distribution to a Benford distribution, even though none of the random variables in the product need be close to Benford at all.

Theorem 44 If X is a continuous random variable, then (X^n) converges in distribution to Benford's law and is Benford with probability one.

Proof See (Berger and Hill 2015, Theorem 8.8). □

Example 45 If U is uniformly distributed on $[0, 1]$, then by Example 10 above, U is not Benford. The sequence of random variables (U, U^2, U^3, \dots) , on the other hand, converges in distribution to Benford's law and is Benford with probability one. In fact, (U^n) converges to Benford's law at rate (n^{-1}) ; see (Berger and Hill 2015, Fig. 1.6).

As a complement to the last theorem, which shows that powers of every continuous random variable converge to Benford's law, the next theorem shows that products of random samples of every continuous random variable also converge to Benford's law.

Theorem 46 If X_1, X_2, \dots are i.i.d. continuous random variables, then the sequence $(X_1, X_1X_2, X_1X_2X_3, \dots)$ converges in distribution to Benford's law and is Benford with probability one.

Proof See (Berger and Hill 2015, Theorem 8.19). □

Example 47 If U_1, U_2, \dots are i.i.d. random variables uniformly distributed on $[0, 1]$, then the sequence of products $U_1, U_1U_2, U_1U_2U_3, \dots$ converges to Benford's law in distribution and is Benford with probability one. In fact, $(U_1U_2 \cdots U_n)$ converges to Benford's law at a rate faster than (2^{-n}) ; see (Berger and Hill 2015, Fig. 8.3).

The next proposition illustrates a curious relationship between the Benford properties of powers of a single distribution and the products of random samples from that distribution.

Proposition 48 Let X_1, X_2, \dots be i.i.d. random variables. If $(X_1, X_1^2, X_1^3, \dots)$ is Benford with probability one, then so is $(X_1, X_1X_2, X_1X_2X_3, \dots)$.

Proof See (Berger and Hill 2015, Corollary 8.21). □

Example 49 Start with any positive number, and multiply repeatedly by either 2 or 3, where the multiplying factor each time is equally likely to be a 2 or a 3, and

independent of the past. The resulting sequence will be Benford with probability one.

To see this, let X_1, X_2, \dots be i.i.d. with $P(X_1 = 2) = P(X_1 = 3) = \frac{1}{2}$. Since the sequences (2^n) and (3^n) are both Benford, the sequence $(X_1^n) = (X_1, X_1^2, X_1^3, \dots)$ is Benford with probability one. By Proposition 48 this implies that the sequence $(X_1, X_1 X_2, X_1 X_2 X_3, \dots)$ is also Benford with probability one, and since Benford sequences are scale-invariant for every $x > 0$, the sequence $(xX_1, xX_1 X_2, xX_1 X_2 X_3, \dots)$ is Benford with probability one.

Note that if X_1, X_2, \dots is a random sample from a distribution that is not Benford, then the classical Glivenko–Cantelli Theorem implies that the empirical distribution converges to the common distribution of the X_k 's, which is *not* Benford. On the other hand, if random samples from *different* distributions are taken in an “unbiased” way, then the empirical distribution of the combined sample will always converge to a Benford distribution. The final theorem in this section identifies a central-limit-like theorem to model this type of convergence to a Benford distribution. Intuitively, it says that when random samples (or data) from different distributions are combined, then, if the different distributions are chosen in an unbiased way, the resulting combined sample will converge to a Benford distribution.

Definition 50 A random probability measure \mathbb{P} is a random variable whose values are probability measures on \mathbb{R} .

Example 51 (1) For a practical realization of a random probability measure \mathbb{P} , simply roll a fair die—if the die comes up 1 or 2, \mathbb{P} is uniformly distributed on $[0, 1]$, and otherwise \mathbb{P} is exponential with mean 1. More formally, let X be a random variable taking values in $\{1, 2, 3, 4, 5, 6\}$ with probability $\frac{1}{6}$ each (e.g., the results of one toss of a fair die). Let P_1 be uniformly distributed on $[0, 1]$, and let P_2 be exponentially distributed with mean 1, i.e., $P_2((-\infty, t]) = 1 - e^{-t}$ for all $t \geq 0$. Define the random probability measure \mathbb{P} by $\mathbb{P} = P_1$ if $X = 1$ or 2, and $\mathbb{P} = P_2$ otherwise. Then with probability $\frac{1}{3}$, the value of \mathbb{P} is a probability measure that is uniformly distributed on $[0, 1]$, and otherwise (i.e., with probability $\frac{2}{3}$), it is a probability measure in \mathbb{R} that is exponential with mean 1; see (Berger and Hill 2015, Example 8.33; Hill 1995).

(2) The classical iterative construction of a random cumulative distribution function by Dubins and Freedman (1967) defines a random probability measure \mathbb{P}_{DF} ; see (Berger and Hill 2015, Example 8.34).

Clearly, some random probability measures will not generate Benford behavior. For example, if \mathbb{P} is P_1 half the time and P_2 half the time, where P_1 is uniformly distributed on $[2, 3]$ and P_2 is uniformly distributed on $[4, 5]$, then random samples from \mathbb{P} will not have any entries with first significant digit 1, and hence cannot be close to Benford.

On the other hand, if a random probability measure is *unbiased* in a sense now to be defined, then it will always lead to Benford behavior. The definition of unbiased below is based on the expected value of \mathbb{P} , that is, the single probability measure

that is the average value of \mathbb{P} . Given a random probability measure \mathbb{P} and any $t \in \mathbb{R}$, the quantity $\mathbb{P}((-\infty, t])$ is a random variable with values between 0 and 1; denote its expected (average) value by $E_{\mathbb{P}}(t)$. It is easy to check that $E_{\mathbb{P}}(t)$ defines (or more precisely, is the cumulative distribution function of) a probability measure $P_{\mathbb{P}}$ on \mathbb{R} , the average probability measure of \mathbb{P} .

Example 52 Let \mathbb{P} be the random probability measure in Example 51(1). Then the average probability measure $P_{\mathbb{P}}$ is the probability distribution of a continuous random variable X with density function $\frac{1}{3} + \frac{2}{3}e^{-x}$ for $0 < x < 1$ and $\frac{2}{3}e^{-x}$ for $x > 1$.

Definition 53 A random probability measure \mathbb{P} has *scale-unbiased significant digits* if its average probability measure $P_{\mathbb{P}}$ has scale-invariant significant digits, and has *base-unbiased significant digits* if $P_{\mathbb{P}}$ has base-invariant significant digits.

Example 54 The classical Dubins-Freedman construction \mathbb{P}_{DF} mentioned in Example 51(2) above has both scale- and base-unbiased significant digits; see (Berger and Hill 2015, Example 8.46).

The next theorem is the key result that shows that if random samples are taken from distributions that are chosen at random in any manner that is unbiased with respect to scale or base, then the resulting empirical distribution of the combined sample always converges in distribution to Benford's law. This may help explain, for example, why the original dataset that Benford drew from many different sources (Benford 1938), why numbers selected at random from newspapers, and why experiments designed to estimate the distribution of leading digits of all numbers on the World Wide Web, all yield results that are close to the logarithmic significant-digit law, i.e., Benford's law.

Theorem 55 Let \mathbb{P} be a random probability measure so that $\mathbb{P}(S \in \{0, 1\}) = 0$ with probability one. Let P_1, P_2, \dots be a random sample (i.i.d. sequence) of probability measures from \mathbb{P} . Fix a positive integer m , and let X_1, X_2, \dots, X_m be a random sample of size m from P_1 , let X_{m+1}, \dots, X_{2m} be a random sample of size m from P_2 , and so on. If \mathbb{P} has scale- or base-unbiased significant digits, then the empirical distribution of the combined sample $X_1, X_2, \dots, X_m, X_{m+1}, \dots$ converges to Benford's law with probability one, that is,

$$P\left(\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : S(X_n) \leq t\}}{N} = \log t \text{ for all } t \in [1, 10)\right) = 1.$$

Proof See (Berger and Hill 2015, Theorem 8.44), noting that slightly different assumptions and notations are used there. \square

Example 56 Since the classical Dubins-Freedman construction \mathbb{P}_{DF} has scale- and base-unbiased significant digits (and has, with probability one, no atoms), by Theorem 55 above, combining random samples from random distributions generated by \mathbb{P}_{DF} will guarantee that the empirical distribution of the combined sample converges to Benford's law.

6 Common errors

The purpose of this section is to familiarize the reader with several recurring errors in the literature on Benford's law, in order that they may be avoided in future research and applications.

Error 1. To be Benford, a random variable or dataset needs to cover at least several orders of magnitude.

As seen in Example 10(2), if U is uniformly distributed on $[0, 1]$, then $X = 10^U$ is exactly Benford, yet X takes only values between 1 and 10.

Error 2. Exponential sequences $(a^n) = (a, a^2, a^3, \dots)$ can generally be assumed to be Benford.

As seen in Example 15, some exponentially increasing sequences such as (2^n) are Benford, and some such as $(10^{n/2})$ are not, so care is needed. Even sequences (a^n) where a is a rational power of 10, although never Benford exactly, may be very close to being Benford depending on a , as can be seen by looking at the sequence $(10^{n/100})$, since $(\langle \frac{n}{100} \rangle)$ is clearly close to being uniformly distributed on $[0, 1]$.

On the other hand *most* exponential sequences are Benford in the sense that if the base number x is selected at random via any continuous distribution, then the sequence (x^n) is Benford with certainty (see Theorem 44), i.e., with probability one.

In contrast to this exponential case, no sequence $(na) = (a, 2a, 3a, 4a, \dots)$ is Benford. Similarly, sequences of sums of i.i.d. random variables with finite variance are *never* Benford, as shown in (Berger and Hill 2015, Theorem 8.30). The authors conjecture that the restriction to distributions with finite variance is not necessary, and that “perhaps even no random walk on the real line at all has Benford paths (in distribution or with probability one)” (Berger and Hill 2015, p. 200).

Error 3. If a distribution or dataset has large spread and is regular, then it is close to Benford.

Unfortunately, this error continues to be widely propagated, likely because it may be traced back to the classical probability text of Feller; see Berger and Hill (2011). As the next example shows, this conclusion does not even hold for the ubiquitous and fundamental normal distribution.

Example 57 If $X = N(7, 1)$ then $P(D_1(X) = 1) \leq 0.00136$, so X is not close to being Benford. Here X is “regular” or “smooth” by almost any criterion, and has standard deviation 1, which may or may not fit the criteria of having a “large spread”. On the other hand, $Y = 100X$ is also regular and has much larger standard deviation than X , but clearly $P(D_1(X) = 1) = P(D_1(Y) = 1)$, so Y is also far from being Benford.

Similarly, no uniform distribution is close to being Benford no matter how spread out it is, and in this case a universal discrepancy between uniform and Benford can be quantified.

Example 58 No uniform random variable is close to Benford's law. In particular, by (Berger and Twelves 2018, Theorem 5.1), if X is a uniform random variable, i.e., X is uniformly distributed on $[a, b]$ for some $a < b$, then for some $1 < t < 10$,

$$|P(S(X) \leq t) - \log t| \geq 0.0758\dots;$$

if $X \geq 0$ or $X \leq 0$ with probability one then the (sharp) numerical bound on the right is even larger, namely 0.134. . . .

Similar bounds away from Benford's law exist for normal and exponential distributions, for example, but for these distributions the corresponding sharp bounds are unknown (Berger and Hill 2015, p. 40).

Error 4. There are relatively simple intuitive arguments to explain Benford's law in general.

For some settings, such as exponentially increasing sequences of constants, fairly simple arguments can be given to show when a sequence is Benford, as was seen in Theorem 14. On the other hand, there is currently no simple intuitive argument to explain the appearance of Benford's law in the wide array of contexts in which it has been observed, including statistics, number theory, dynamical systems, and real-world data. More concretely, there is no theory at all, let alone a simple one, even to decide whether the sequence (1, 2, 5, 26, 677, . . .) starting with 1 and proceeding by squaring the last number and adding 1, is Benford or not; see Example 38(1). The interested reader is referred to Berger and Hill (2011) for a more detailed treatise on the difficulty of finding an easy explanation of Benford's law.

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References

- Benford F (1938) The law of anomalous numbers. *Proc Am Philos Soc* 78(4):551–572
- Berger A, Hill TP (2011) Benford's law strikes back: no simple explanation in sight for mathematical gem. *Math Intell* 33(1):85–91
- Berger A, Hill TP (2015) An introduction to Benford's law. Princeton University Press, Princeton
- Berger A, Hill TP, Rogers E (2009) Benford Online Bibliography. www.benfordonline.net. Accessed 17 April 2020.
- Berger A, Twelves I (2018) On the significands of uniform random variables. *J Appl Probab* 55:353–367
- Dubins L E, Freedman D A (1967) Random distribution functions. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66)*, Vol. II: Contributions to Probability Theory, Part 1, pp. 183–214. University of California Press, Berkeley
- Hill TP (1995) A statistical derivation of the significant-digit law. *Statist Sci* 10(4):354–363
- Miller SJ (ed) (2015) Benford's Law: theory and applications. Princeton University Press, Princeton
- Newcomb S (1881) Note on the frequency of use of the different digits in natural numbers. *Am J Math* 4(1–4):39–40
- Nigrini MJ (2012) Benford's law: applications for forensic accounting, auditing, and fraud detection. Wiley, Hoboken

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