An improved Maximum Allowable Transfer Interval for L^p-stability of Networked Control Systems

Arnulf Jentzen, Frank Leber, Daniela Schneisgen, Arno Berger, Stefan Siegmund

***Abstract*—An elementary self-contained proof is given for an improved bound on the maximum allowable transfer interval that guarantees L^p-stability in networked control systems with disturbances.**

***Index Terms*—L^p-stability, networked control systems.**

I. Introduction

A *Networked Control System (NCS)* consists of multiple feedback control loops sharing a serial communication channel. When compared with traditional multi-channel control, the NCS architecture has the advantages of low cost, easy maintenance and great flexibility. As a consequence, analysis and design of NCS have received a lot of attention lately, as evidenced for instance by [1], [2], [5] and the many references therein. A key feature of NCS is that, due to the reliance on a single channel, the overall system performance and stability may deteriorate if the communication is overly delayed or infrequent. An important problem in the analysis of NCS, therefore, is to find rigorous yet practicable bounds for the time span between transmission times up to which stability of the whole system can be guaranteed. Substantial progress has been made recently in determining this *maximum allowable transfer interval* efficiently. The purpose of the present note is to further improve one pivotal result in this regard for a special class of network protocols: In the same set-up as [2], the main result (Theorem 1) provides an upper bound on the transfer interval that is universally larger than the one developed in that paper. While the quantitative improvement is modest, the argument by which it is achieved is short and elementary and thus may be useful for any future work on the subject. Though similar in spirit, the corresponding results in [1], [5], are based on somewhat different assumptions and are not immediately comparable to the result presented here.

II. Networked Control Systems

Consider an *NCS* as described in [2], allowing for jumps and disturbances. The network's *transmission times* are $(\theta_j)_{j\in\mathbb{N}_0}$ with $\theta_0 = 0$ and $\varepsilon \le \theta_{j+1} - \theta_j \le \tau$ for all $j \in \mathbb{N}_0$, where $0 < \varepsilon < \tau$. Note that $\varepsilon > 0$ is arbitrary, its sole purpose being to rule out solutions with infinitely many jumps in finite time. The bound τ is referred to as the *maximum allowable transfer interval (MATI)*. At each transmission time θ_j the protocol gives access to the communication network to one of the internal nodes. The structure of a general NCS is depicted schematically in Fig. 1. At each transmission time θ_j the output signals u and y of, respectively, controller and plant are transmitted via the network, thus providing the input signals $\widehat{u}$ and $\widehat{y}$ to plant and controller. Between transmission times, $\widehat{u}$ and $\widehat{y}$ obey an intrinsic dynamics governed by $\widehat{f}_C$ and $\widehat{f}_P$, respectively; in the simplest case, $\widehat{f}_C = \widehat{f}_P = 0$, implying that $\widehat{u}$, $\widehat{y}$ are constant between transmission times. The entire system is subject to the external perturbation w; throughout it will be assumed that w is locally integrable. As detailed in [2], upon introduction of the combined state $x = \begin{bmatrix} x_P \\ x_C \end{bmatrix}$ and network error $e = \begin{bmatrix} \widehat{y} - y \\ \widehat{u} - u \end{bmatrix}$ the equations governing the NCS according to Fig. 1 can be written concisely as

$$\dot{x} = f(t, x, e, w)\,, \tag{1.1}$$

$$\dot{e} = g(t, x, e, w)\,, \tag{1.2}$$

$$e(\theta_j) = h\big(j, e(\theta_j^-)\big)\,, \tag{1.3}$$

where $f : \mathbb{R}_{\ge 0} \times \mathbb{R}^{n_x+n_e+n_w} \to \mathbb{R}^{n_x}$ and $g : \mathbb{R}_{\ge 0} \times \mathbb{R}^{n_x+n_e+n_w} \to \mathbb{R}^{n_e}$, $h : \mathbb{N}_0 \times \mathbb{R}^{n_e} \to \mathbb{R}^{n_e}$. System (1) is hybrid in that it combines the differential equations (1.1), (1.2) for x, e with the difference (jump) equation (1.3) for e at transmission times. The function h is a key ingredient of (1): it encodes the network protocol by specifying how at any transmission time access to the network is granted to different nodes in the system.

Given $x_0 \in \mathbb{R}^{n_x}$, $e_0 \in \mathbb{R}^{n_e}$, a solution of the initial value problem

$$x(0) = x_0\,, \quad e(0) = e_0\,, \tag{2}$$

for (1) is understood to be any pair (x, e) of functions satisfying (2) such that (1.1) and (1.2) hold for almost all $t > 0$, and (1.3) holds for all $j \in \mathbb{N}$; implicit in the latter is that $x : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n_x}$ and $e : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n_e}$ are absolutely continuous on, respectively, $\mathbb{R}_{\ge 0}$ and $[\theta_j, \theta_{j+1}[$ for every $j \in \mathbb{N}$, and $e(\theta_j^-) := \lim_{\delta \searrow 0} e(\theta_j - \delta)$ exists. Here any function F on $[a, b[$ with $a < b$ will be called absolutely continuous if $F|_{[a,c]}$ is absolutely continuous for every $a < c < b$. Throughout it will be assumed that f, g, and the functions $f_C, \widehat{f}_C, f_P, \widehat{f}_P, g_C, g_P$ in the original NCS (see [2] for details), are sufficiently smooth and regular for (1),(2) to have a unique solution for each x_0, e_0. Also, a measurable function $H : \mathbb{R}^{n_x} \to \mathbb{R}^l$ from the state space $\mathbb{R}^{n_x}$ to $\mathbb{R}^l$ will be considered, which models the output of the hybrid system (1).

For ease of presentation, denote by $\mathbb{R}_{\ge 0}$ the set of all non-negative real numbers, and for every $t \in \mathbb{R}_{\ge 0}$ let $\langle t \rangle := \max\{j \in \mathbb{N}_0 : \theta_j \le t\}$ so that $\langle t \rangle = j$ if and only if $t \in [\theta_j, \theta_{j+1}[$, and $t \ge \theta_{\langle t \rangle}$ for all $t \in \mathbb{R}_{\ge 0}$. For every $y \in L^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{R}^d)$ with $d \in \mathbb{N}$ and compact interval $I \subset \mathbb{R}_{\ge 0}$, the norm $\|y\|_{L^p(I)}$ with $p \ge 1$ is given by $\left(\int_I \|y(t)\|^p \,\mathrm{d}t\right)^{1/p}$ where $\|\cdot\|$ is any norm on $\mathbb{R}^d$. Throughout, if $p > 1$ then $q > 1$ denotes the unique number with $1/p + 1/q = 1$.

The main goal of this article is to establish conditions ensuring that the system (1) is L^p-stable from w to $H(x)$, which means that there exist constants $K, \gamma \ge 0$ independent of $x_0 \in \mathbb{R}^{n_x}$, $e_0 \in \mathbb{R}^{n_e}$ and w, such that

$$\|H(x)\|_{L^p[0,t]} \le K(\|x_0\| + \|e_0\|) + \gamma \|w\|_{L^p[0,t]}$$

holds for all $t \ge 0$.

III. A stability theorem for NCS

An important problem in the study of the NCS (1) is to identify stability criteria. Naturally, any such criterion has to take into account

A. Jentzen and D. Schneisgen are with the Institut für Mathematik, Johann Wolfgang Goethe Universität Frankfurt a. Main, Germany (email: {jentzen, schneisg}@math.uni-frankfurt.de).

F. Leber and S. Siegmund are with the Fachrichtung Mathematik, Institut für Analysis, TU Dresden, Germany (email: Frank_Leber@t-online.de, siegmund@tu-dresden.de).

A. Berger is with the Department of Mathematical and Statistical Sciences, University of Alberta, Canada (email: berger@ualberta.ca).

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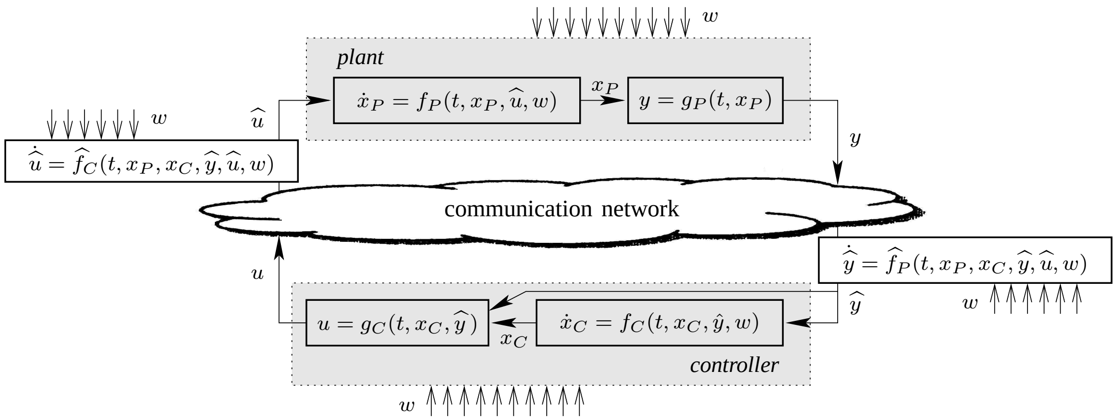

Fig. 1: A schematic model of the general NCS structure.

three main aspects: the properties of the protocol, as described by (1.3); the intrinsic dynamics of plant and controller, as given by (1.1); and the maximal time between transmissions, as measured by τ. In accordance with [2], the following assumptions on, respectively, the protocol and the intrinsic dynamics are made with an appropriately chosen measurable function $H : \mathbb{R}^{n_x} \to \mathbb{R}^l$.

(AV) There exists a function $V : \mathbb{N}_0 \times \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ such that $V(j, \cdot) : \mathbb{R}^{n_e} \to \mathbb{R}_{\geq 0}$ is locally Lipschitz (and hence almost everywhere differentiable) for every $j \in \mathbb{N}_0$, and there exist positive constants $B_1, L > 0$, as well as $0 < \rho < 1$, such that

(i) $B_1^{-1}\|e\| \leq V(j, e) \leq B_1\|e\|$ for all $(j, e) \in \mathbb{N}_0 \times \mathbb{R}^{n_e}$;

(ii) For almost all $t \geq 0$ and almost all $e \in \mathbb{R}^{n_e}$, and for all $(j, x, w) \in \mathbb{N}_0 \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}$

$$\nabla_e V(j, e) \cdot g(t, x, e, w) \leq LV(j, e) + \|H(x)\| + B_1\|w\| \,;$$

(iii) $V\big(j + 1, h(j, e)\big) \leq \rho V(j, e)$ for all $(j, e) \in \mathbb{N}_0 \times \mathbb{R}^{n_e}$.

(Ap) System (1.1) is L^p-stable from (V, w) to $H(x)$ with V as in (AV), that is, for some $B_2, \gamma > 0$

$$\|H(x)\|_{L^p[0,t]} \leq B_2(\|x_0\| + \|w\|_{L^p[0,t]}) + \gamma\|V(\langle\cdot\rangle, e)\|_{L^p[0,t]}$$

for all $t > 0$.

The assumptions (AV) and (Ap) are naturally met in many situations of practical importance. If so, the overall stability of (1) crucially depends on the transmission times not being too far apart. The following theorem is the main result of this note.

Theorem 1. Assume that (1) satisfies (AV) and (Ap) for some $p > 1$. Then (1) is L^p-stable from w to $(H(x), e)$, i.e. with some constant C,

$$\|H(x)\|_{L^p[0,t]} + \|e\|_{L^p[0,t]} \leq C(\|x_0\| + \|e_0\| + \|w\|_{L^p[0,t]}) \quad (3)$$

holds for all $t \geq 0$, provided that $\tau < \tau_{\mathit{MATI}}$, where τ_{MATI} is the unique zero of

$$F_{\gamma,L,p,\rho}(\tau) := \gamma \left(\frac{e^{Lp\tau} - 1}{Lp}\right)^{1/p} \left(\frac{e^{Lq\tau} - 1}{Lq}\right)^{1/q} \frac{\rho}{1 - \rho e^{L\tau}} + {} + \gamma \frac{e^{L\tau} - 1}{Lp^{1/p}} - 1$$

in the interval $]0, -L^{-1} \ln \rho[$.

Remark 2. Since $\tau \mapsto F_{\gamma,L,p,\rho}(\tau)$ is smooth and strictly increasing, τ_{MATI} can easily be determined numerically. For $p = 2$ a short computation confirms the explicit formula

$$L\,\tau_{\mathit{MATI}} = \ln L \left\{ \tfrac{1+\sqrt{2}}{\gamma} + \tfrac{(2+\sqrt{2})(1+\rho)}{2L\rho} - \sqrt{\tfrac{3+2\sqrt{2}}{\gamma^2} + \tfrac{4+3\sqrt{2}}{\gamma L} + \tfrac{2+\sqrt{2}}{\gamma L\rho} + \tfrac{1+\sqrt{2}}{L^2\rho} + \tfrac{3+2\sqrt{2}}{2L^2\rho^2} + \tfrac{1}{2L^2}} \right\}$$

Under the identical assumptions as in Theorem 1, stability is established in [2] for

$$\tau < \tau_{[2]} = \frac{1}{L} \ln \frac{L + \gamma}{\rho L + \gamma} \,. \quad (4)$$

That τ_{MATI} is universally better (i.e. larger) than $\tau_{[2]}$ is the content of

Corollary 3. Under the assumptions of Theorem 1, and with $\tau_{[2]}$ given by (4),

$$\tau_{[2]} < \tau_{\mathit{MATI}} \quad (5)$$

holds for every $\gamma, L > 0$, $p > 1$ and $0 < \rho < 1$.

Example 4 below exemplifies (5) in the benchmark example of a batch reactor. Using somewhat different concepts and techniques, in [1] stability has been established for

$$\tau \leq \tau_{[1]} = \frac{1}{\sqrt{\gamma^2 - L^2}} \arctan\left(\frac{(1 - \rho^2)\sqrt{\gamma^2 - L^2}}{2\rho(\gamma - L) + L(1 + \rho)^2}\right) \,, \quad (6)$$

and in [5] for some specific class of protocols and

$$\tau \leq \tau_{[5]} = \frac{\ln z}{NL} \,, \quad (7)$$

where N denotes the number of nodes, and $z > 0$ satisfies $z\,(L + \gamma N) - \gamma N z^{1-1/N} - 2L = 0$. While for the batch reactor example $\tau_{[1]}, \tau_{[5]} > \tau_{\mathit{MATI}}$, the authors do not know whether such a relation holds generally. Moreover, unlike [1, Thm.1] Theorem 1 allows for disturbances w. On the other hand, results in [1] can be easily extended to some classes of perturbed systems, in particular to the system considered in Example 4. Note also that [5, Thm.5.2] assumes the protocol to be *uniformly persistently exciting*, which is not the case e.g. for the TOD protocol.

Example 4. The linearized model of an unstable batch reactor is considered in [1], [2], [4], [5] as a benchmark problem. When written in the standard form (1), the governing equations of this two-input-two-output system are

$$\dot{x} = A_{11}x + A_{12}e + w_x \,,$$
$$\dot{e} = A_{21}x + A_{22}e + w_e \,,$$

with $w_x \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^6)$, $w_e \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R}^4)$, and constant matrices $A_{11}, A_{12}, A_{21}, A_{22}$ according to

$$A_{11} = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 & 0 & 0 \\ -0.5914 & -15.65 & 0 & 0.675 & -11.36 & 0 \\ -14.66 & 2.001 & -22.38 & 21.62 & -2.272 & -25.17 \\ 0.048 & 2.001 & 1.343 & -2.104 & -2.272 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -11.36 & 0 & 0 \\ -15.73 & -2.272 & 0 & 0 \\ 0 & -2.272 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 13.33 & 0.2077 & 17.01 & -18.05 & 0 & 25.17 \\ 0.5914 & 15.65 & 0 & -0.675 & 11.36 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 15.73 & 0 & 0 & 0 \\ 0 & 11.36 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The jump relation (1.3) is determined by the protocol. Two popular choices are the so-called *round robin (RR)* and *try-once-discard (TOD)* protocols. The RR protocol is static in that it gives access to the two internal nodes[1] according to a pre-set algorithm that is independent of e, concretely,

$$h_{\scriptscriptstyle RR}(j,e) = \begin{cases} \mathrm{diag}[1,0,0,0]\, e & \text{if } j \text{ is even,} \\ \mathrm{diag}[0,1,0,0]\, e & \text{if } j \text{ is odd;} \end{cases}$$

the fact that $h_{\scriptscriptstyle RR}(j,e)_3 = h_{\scriptscriptstyle RR}(j,e)_4 = 0$ for all j, e reflects the underlying model assumption that only the plant output signal y is transmitted via the network whereas $\widehat{u} = u$, i.e., the controller output signal is transmitted to the plant through a separate perfect channel. The dynamic TOD protocol grants access depending on the actual value of e, concretely

$$h_{\scriptscriptstyle TOD}(j,e) = \begin{cases} \mathrm{diag}[1,0,0,0]\, e & \text{if } |e_2| \geq |e_1|, \\ \mathrm{diag}[0,1,0,0]\, e & \text{if } |e_2| < |e_1|. \end{cases}$$

Thus the node showing the greater error is given access at θ_j^-, and the corresponding component of $e(\theta_j)$ equals zero.

For Theorem 1 to apply, assumptions (AV) and (Ap) must hold. For L^2-stability, i.e. for $p = 2$, it is demonstrated in [2] that this is indeed the case with a function V constructed from h, with $L = 15.73$, $\rho = 1/\sqrt{2}$, and with $\gamma = 22.52$, $H(x) = \sqrt{2}A_{21}x$ in the RR case and $\gamma = 15.92$, $H(x) = A_{21}x$ in the TOD case, respectively. Using these data, Table I shows the values of $\tau_{[2]}, \tau_{MATI}$ and $\tau_{[1]}$ for both protocols, and of $\tau_{[5]}$ for the RR protocol. For the batch reactor example the relative increase of τ_{MATI} over $\tau_{[2]}$ is 7.2% for the RR and 6.5% for the TOD protocol.

	$\tau_{[2]}$	τ_{MATI}	$\tau_{[1]}$	$\tau_{[5]}$
RR	$8.159 \cdot 10^{-3}$	$8.750 \cdot 10^{-3}$	$8.956 \cdot 10^{-3}$	$1.052 \cdot 10^{-2}$
TOD	$1.0 \cdot 10^{-2}$	$1.065 \cdot 10^{-2}$	$1.084 \cdot 10^{-2}$	N/A

TABLE I: Comparison of the theoretical upper bounds on τ for L^2-stability as given by (4), Theorem 1, (6) and (7), respectively.

Figure 2 graphs the theoretical upper bounds on τ for L^p-stability as given by Theorem 1 and (4), respectively, as a function of $1 < p < \infty$ with parameters $L = 15.73$, $\rho = \frac{1}{\sqrt{2}}$ for the batch reactor with TOD protocol. Figure 3 compares the various theoretical upper bounds on τ depending on $\gamma \in [10, 100]$ with parameters $L = 15.73$, $N = 2$, $\rho = \frac{1}{\sqrt{2}}$, $H(x) = \sqrt{2}A_{21}x$ for the batch reactor with RR protocol.

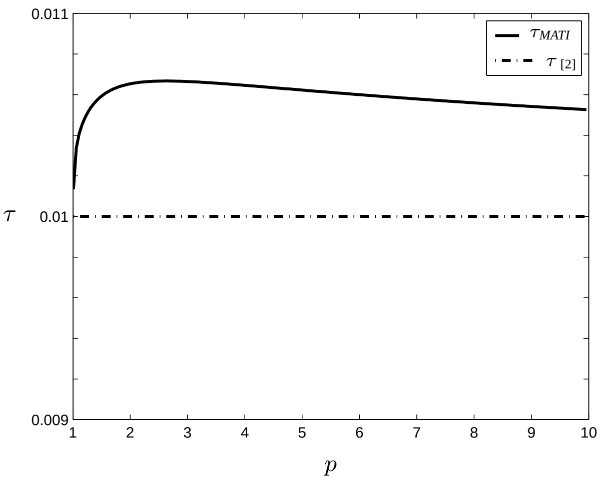

Fig. 2: Comparison of the theoretical upper bounds on τ for L^p-stability as given by Theorem 1 and (4), respectively, as a function of p, using the TOD protocol for the batch reactor example.

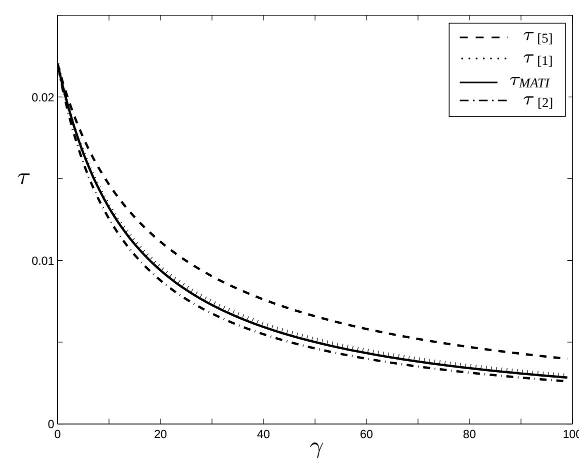

Fig. 3: Graphing the theoretical upper bounds on τ for L^2-stability and varying γ, as given by (7), (6), Theorem 1, and (4), respectively, for the batch reactor example.

IV. Proofs

The proof of Theorem 1 rests upon two lemmas that are of interest on their own. Lemma 5 is a *generalisation to impulsive systems* of the *classical Gronwall inequality*. Lemma 6 provides an expedient estimate for a piecewise expression arising from Lemma 5 that is instrumental in the proof of the main result.

Lemma 5. Let $y, z \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$ be non-negative and assume that, for every $j \in \mathbb{N}_0$, the function z is absolutely continuous on $[\theta_j, \theta_{j+1}[$ and $z(\theta_{j+1}^-) = \lim_{\delta \searrow 0} z(\theta_{j+1} - \delta)$ exists. If, with some constants $L, \rho \geq 0$,

(i) $\dfrac{dz}{dt}(t) \leq Lz(t) + y(t)$ for almost all $t \geq 0$, and
(ii) $z(\theta_{j+1}) \leq \rho z(\theta_{j+1}^-)$ for all $j \in \mathbb{N}_0$,

[1]Nodes are referred to as *links* in [5]; in the present example, $N = 2$.

then the inequality

$$z(t) \le e^{Lt}\rho^{\langle t\rangle} z(0) + \int_0^t e^{L(t-s)}\rho^{\langle t\rangle-\langle s\rangle} y(s)\,\mathrm{d}s \tag{8}$$

holds for all $t \ge 0$. If equality holds in (i) and (ii), then equality holds in (8) as well.

Proof: For all $t \in [\theta_j, \theta_{j+1}[$ and $j \in \mathbb{N}_0$, assumption (i) and the classical Gronwall lemma imply that

$$z(t) \le e^{L\left(t-\theta_j\right)} z(\theta_j) + \int_{\theta_j}^t e^{L(t-s)} y(s)\,\mathrm{d}s. \tag{9}$$

In particular, by (9) and assumption (ii),

$$z(\theta_j) \le \rho z(\theta_j^-) \le e^{L(\theta_j-\theta_{j-1})}\rho z(\theta_{j-1}) + \rho\int_{\theta_{j-1}}^{\theta_j} e^{L(\theta_j-s)} y(s)\,\mathrm{d}s\,,$$

and hence by induction

$$z(\theta_j) \le e^{L\theta_j}\rho^j z(0) + \sum\nolimits_{k=1}^{j}\rho^{j-k+1}\int_{\theta_{k-1}}^{\theta_k} e^{L(\theta_j-s)} y(s)\,\mathrm{d}s\,,$$

which, together with (9) and $\langle s\rangle = k-1$ for every $s \in [\theta_{k-1}, \theta_k[$, shows that

$$\begin{aligned} z(t) &\le e^{Lt}\rho^j z(0) + \sum\nolimits_{k=1}^{j}\rho^{j-k+1}\int_{\theta_{k-1}}^{\theta_k} e^{L(t-s)} y(s)\,\mathrm{d}s\, + \\ &\qquad + \int_{\theta_j}^t e^{L(t-s)} y(s)\,\mathrm{d}s \\ &= e^{Lt}\rho^{\langle t\rangle} z(0) + \int_0^t e^{L(t-s)}\rho^{\langle t\rangle-\langle s\rangle} y(s)\,\mathrm{d}s \end{aligned}$$

holds for all $t \in [\theta_j, \theta_{j+1}[$. Since $j \in \mathbb{N}_0$ was arbitrary, this proves (8). ■

Lemma 6. Assume that the numbers $\tau, L > 0$ and $0 < \rho < 1$ satisfy $\tau < -L^{-1}\ln\rho$. Given $y \in L^1_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{R}^d)$, define functions $Y_1, Y_2 : \mathbb{R}_{\ge 0} \to \mathbb{R}^d$ as

$$Y_1(t) := \int_{\theta_{\langle t\rangle}}^t e^{L(t-s)} y(s)\,\mathrm{d}s\,, \quad Y_2(t) := \int_0^t e^{L(t-s)}\rho^{\langle t\rangle-\langle s\rangle} y(s)\,\mathrm{d}s\,.$$

For every $p > 1$, if $y \in L^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{R}^d)$ then $Y_1, Y_2 \in L^p_{\mathrm{loc}}(\mathbb{R}_{\ge 0}; \mathbb{R}^d)$, and for all $t \ge 0$

$$\|Y_1\|_{L^p[0,t]} \le \lambda_1 \|y\|_{L^p[0,t]}\,, \quad \|Y_2\|_{L^p[0,t]} \le \lambda_2 \|y\|_{L^p[0,t]}\,, \tag{10}$$

with

$$\begin{aligned} \lambda_1 &= \frac{e^{L\tau}-1}{Lp^{1/p}}\,, \\ \lambda_2 &= \left(\frac{e^{Lp\tau}-1}{Lp}\right)^{1/p}\left(\frac{e^{Lq\tau}-1}{Lq}\right)^{1/q}\frac{\rho}{1-\rho e^{L\tau}} + \lambda_1\,. \end{aligned} \tag{11}$$

Proof: For all $t \ge 0$, Hölder's inequality implies that

$$\begin{aligned} \|Y_1(t)\|^p &\le \left(\int_{\theta_{\langle t\rangle}}^t e^{L(t-s)}\|y(s)\|\,\mathrm{d}s\right)^p \\ &\le \left(\int_{\theta_{\langle t\rangle}}^t e^{L(t-s)}\,\mathrm{d}s\right)^{p-1}\int_{\theta_{\langle t\rangle}}^t e^{L(t-s)}\|y(s)\|^p\,\mathrm{d}s \\ &= L^{1-p}\left(e^{L(t-\theta_{\langle t\rangle})}-1\right)^{p-1}\int_{\theta_{\langle t\rangle}}^t e^{L(t-s)}\|y(s)\|^p\,\mathrm{d}s\,, \end{aligned}$$

and consequently, with $j = \langle t\rangle$,

$$\begin{aligned} &L^p\int_{\theta_j}^t \|Y_1(s)\|^p\mathrm{d}s \le \\ &\le L\int_{\theta_j}^t\left(e^{L(s-\theta_j)}-1\right)^{p-1}\int_{\theta_j}^s e^{L(s-\sigma)}\|y(\sigma)\|^p\,\mathrm{d}\sigma\,\mathrm{d}s \\ &= L\int_{\theta_j}^t\left(\int_\sigma^t\left(e^{L(s-\theta_j)}-1\right)^{p-1}e^{Ls}\mathrm{d}s\right)e^{-L\sigma}\|y(\sigma)\|^p\mathrm{d}\sigma \\ &= p^{-1}\int_{\theta_j}^t\left((e^{L(t-\theta_j)}-1)^p - (e^{L(\sigma-\theta_j)}-1)^p\right)e^{L(\theta_j-\sigma)}\|y(\sigma)\|^p\mathrm{d}\sigma \\ &\le \frac{(e^{L\tau}-1)^p}{p}\int_{\theta_j}^t\|y(s)\|^p\mathrm{d}s\,. \end{aligned}$$

In particular therefore

$$\int_{\theta_k}^{\theta_{k+1}}\|Y_1(s)\|^p\,\mathrm{d}s \le \frac{(e^{L\tau}-1)^p}{L^p p}\left(\|y\|_{L^p[\theta_k,\theta_{k+1}]}\right)^p\,, \quad \forall k \in \mathbb{N}_0\,,$$

and summation over k yields the claimed bound on Y_1.

To prove the estimate for Y_2, note first that, by Hölder's inequality and $\theta_{j+1}-\theta_j \le \tau$,

$$\begin{aligned} &\|Y_2(t) - Y_1(t)\| \le \int_0^{\theta_{\langle t\rangle}} e^{L(t-s)}\rho^{\langle t\rangle-\langle s\rangle}\|y(s)\|\,\mathrm{d}s \\ &= \sum\nolimits_{j=0}^{\langle t\rangle-1}\rho^{\langle t\rangle-j}\int_{\theta_j}^{\theta_{j+1}} e^{L(t-s)}\|y(s)\|\,\mathrm{d}s \\ &\le \sum\nolimits_{j=0}^{\langle t\rangle-1}\rho^{\langle t\rangle-j}\left(\int_{\theta_j}^{\theta_{j+1}} e^{Lq(t-s)}\,\mathrm{d}s\right)^{1/q}\left(\int_{\theta_j}^{\theta_{j+1}}\|y(s)\|^p\,\mathrm{d}s\right)^{1/p} \\ &\le \sum\nolimits_{j=0}^{\langle t\rangle-1}\rho^{\langle t\rangle-j}e^{L(t-\theta_j)}L^{-1/q}q^{-1/q}(1-e^{-Lq\tau})^{1/q}\|y\|_{L^p[\theta_j,\theta_{j+1}]} \\ &\le e^{L(t-\theta_{\langle t\rangle})}\left(\frac{1-e^{-Lq\tau}}{Lq}\right)^{1/q}\sum\nolimits_{j=0}^{\langle t\rangle-1}(\rho e^{L\tau})^{\langle t\rangle-j}\|y\|_{L^p[\theta_j,\theta_{j+1}]} \\ &= \left(\frac{1-e^{-Lq\tau}}{Lq}\right)^{1/q} Z(t)\,, \end{aligned}$$

with the auxiliary function

$$Z(t) = e^{L(t-\theta_{\langle t\rangle})}\sum\nolimits_{j=0}^{\langle t\rangle-1}(\rho e^{L\tau})^{\langle t\rangle-j}\|y\|_{L^p[\theta_j,\theta_{j+1}]}\,.$$

A bound on $\|Z\|_{L^p[0,t]}$ follows from

$$\begin{aligned} &\int_0^t Z(s)^p\mathrm{d}s = \\ &= \int_0^t e^{Lp(s-\theta_{\langle s\rangle})}\left(\sum\nolimits_{j=0}^{\langle s\rangle-1}(\rho e^{L\tau})^{\langle s\rangle-j}\|y\|_{L^p[\theta_j,\theta_{j+1}]}\right)^p\mathrm{d}s \\ &\le \sum\nolimits_{k=0}^{\langle t\rangle}\int_{\theta_k}^{\theta_{k+1}} e^{Lp(s-\theta_k)}\left(\sum\nolimits_{j=0}^{k-1}(\rho e^{L\tau})^{k-j}\|y\|_{L^p[\theta_j,\theta_{j+1}]}\right)^p\mathrm{d}s \\ &\le \frac{e^{Lp\tau}-1}{Lp}\sum\nolimits_{k=0}^{\langle t\rangle}\left(\sum\nolimits_{j=0}^{k-1}(\rho e^{L\tau})^{k-j}\|y\|_{L^p[\theta_j,\theta_{j+1}]}\right)^p\,, \end{aligned}$$

together with the observation that, since $\rho e^{L\tau} < 1$,

$$\begin{aligned} &\sum\nolimits_{k=0}^{\langle t\rangle}\left(\sum\nolimits_{j=0}^{k-1}(\rho e^{L\tau})^{k-j}\|y\|_{L^p[\theta_j,\theta_{j+1}]}\right)^p \\ &\le \sum\nolimits_{k=0}^{\langle t\rangle}\left(\sum\nolimits_{j=0}^{k-1}(\rho e^{L\tau})^{k-j}\right)^{p-1}\sum\nolimits_{j=0}^{k-1}(\rho e^{L\tau})^{k-j}\|y\|^p_{L^p[\theta_j,\theta_{j+1}]} \\ &\le \left(\frac{\rho e^{L\tau}}{1-\rho e^{L\tau}}\right)^{p-1}\sum\nolimits_{k=0}^{\langle t\rangle}\sum\nolimits_{j=0}^{k-1}(\rho e^{L\tau})^{k-j}\|y\|^p_{L^p[\theta_j,\theta_{j+1}]} \\ &= \left(\frac{\rho e^{L\tau}}{1-\rho e^{L\tau}}\right)^{p-1}\sum\nolimits_{j=0}^{\langle t\rangle-1}\left(\sum\nolimits_{k=j+1}^{\langle t\rangle}(\rho e^{L\tau})^{k-j}\right)\|y\|^p_{L^p[\theta_j,\theta_{j+1}]} \end{aligned}$$

$$\le \left(\frac{\rho e^{L\tau}}{1-\rho e^{L\tau}}\right)^p \|y\|^p_{L^p[0,\theta_{\langle t\rangle}]} ,$$

so that overall

$$\|Z\|_{L^p[0,t]} \le \left(\frac{e^{Lp\tau}-1}{Lp}\right)^{1/p} \left(\frac{\rho e^{L\tau}}{1-\rho e^{L\tau}}\right) \|y\|_{L^p[0,t]} .$$

This implies that, for all $t \ge 0$,

$$\|Y_2\|_{L^p[0,t]} \le \left(\frac{1-e^{-Lq\tau}}{Lq}\right)^{1/q} \|Z\|_{L^p[0,t]} + \|Y_1\|_{L^p[0,t]} ,$$

and proves the bound claimed for Y_2. ■

Proof of Theorem 1: Note first that with the notation of Lemma 6, $F_{\gamma,L,p,\rho} = \gamma\lambda_2 - 1$. Given $\gamma, L > 0$, $p > 1$, and $0 < \rho < 1$, the map $F_{\gamma,L,p,\rho}$ is continuous and strictly increasing on $[0, -L^{-1}\ln\rho[$ with $F_{\gamma,L,p,\rho}(0) = -1$ and $\lim_{\tau\uparrow -L^{-1}\ln\rho} F_{\gamma,L,p,\rho}(\tau) = +\infty$. Thus τ_{MATI}, as a zero of $F_{\gamma,L,p,\rho}$, is uniquely determined. Assume that $\tau < \tau_{\mathit{MATI}}$, hence $\gamma\lambda_2 < 1$, and define auxiliary functions $y, z : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ according to

$$z(t) := V\big(\langle t\rangle, e(t)\big) , \quad y(t) := \|H(x(t))\| + B_1\|w(t)\| .$$

For these functions, assumptions (AV,ii), with [6, Corollary, p.64], and (AV,iii) imply that

$$\frac{dz}{dt}(t) \le Lz(t) + y(t) , \quad z(\theta_{j+1}) \le \rho z(\theta^-_{j+1}) ,$$

for almost all $t \ge 0$ and $j \in \mathbb{N}_0$, respectively. So Lemma 5 applies and shows in turn that

$$z\,(t) \le e^{Lt}\rho^{\langle t\rangle} z\,(0) + \int_0^t e^{L(t-s)}\rho^{\langle t\rangle - \langle s\rangle} y(s)\,\mathrm{d}s ,$$

and therefore, by Lemma 6, for all $t \ge 0$,

$$\begin{aligned}\|z\|_{L^p[0,t]} &\le C_0 z(0) + \lambda_2\|y\|_{L^p[0,t]} \\ &\le B_1(C_0+\lambda_2)(\|e_0\| + \|w\|_{L^p[0,t]}) + \lambda_2\|H(x)\|_{L^p[0,t]} ,\end{aligned}$$

with an appropriate C_0 not depending on t. From assumption (Ap), it follows that

$$\begin{aligned}\|H(x)\|_{L^p[0,t]} &\le B_2(\|x_0\| + \|w\|_{L^p[0,t]}) + \gamma\|z\|_{L^p[0,t]} \\ &\le B(1+\gamma C_0+\gamma\lambda_2)(\|x_0\| + \|e_0\| + \|w\|_{L^p[0,t]}) + \\ &\quad + \gamma\lambda_2\|H(x)\|_{L^p[0,t]} ,\end{aligned}$$

for all $t \ge 0$, where $B = \max\{B_1, B_2\}$, and since $\gamma\lambda_2 < 1$,

$$\|H(x)\|_{L^p[0,t]} \le B\,\frac{1+\gamma C_0+\gamma\lambda_2}{1-\gamma\lambda_2}\,(\|x_0\| + \|e_0\| + \|w\|_{L^p[0,t]}) ,$$

for all $t \ge 0$. By (AV,i), $\|e\|_{L^p[0,t]} \le B\|z\|_{L^p[0,t]}$, and hence the proof is complete. ■

The proof of Corollary 3 will make use of the following elementary fact.

Lemma 7. Let $p > 1$. Then

$$\left(\frac{x^p-1}{p}\right)^{1/p} \left(\frac{x^q-1}{q}\right)^{1/q} \le x(x-1) , \quad \forall x \ge 1 . \tag{12}$$

Proof: For any $\alpha \ge 1$ define $\varphi_\alpha(x) := \alpha x^{\alpha-1}(x-1) - x^\alpha + 1$ and observe that $\varphi_\alpha(x) \ge 0$ for all $x \ge 1$ because $\varphi_\alpha(1) = 0$ and $\varphi'_\alpha(x) = \alpha(\alpha-1)x^{\alpha-2}(x-1) \ge 0$. Consequently,

$$\left(\frac{x^\alpha-1}{\alpha}\right)^{\frac{1}{\alpha}} \le x^{\left(1-\frac{1}{\alpha}\right)}\,(x-1)^{\frac{1}{\alpha}}$$

for all $\alpha \ge 1$ and $x \ge 1$. This yields

$$\begin{aligned}\left(\frac{x^p-1}{p}\right)^{1/p}\left(\frac{x^q-1}{q}\right)^{1/q} &\le x^{1-1/p}(x-1)^{1/p}x^{1-1/q}(x-1)^{1/q} \\ &= x(x-1) , \quad \forall x \ge 1 .\end{aligned}$$

Note that (12) becomes an equality as $p \to 1$ or $p \to +\infty$. ■

Proof of Corollary 3: The number $\tau_{[2]}$ is the unique zero of $G_{\gamma,L,\rho}(\tau) = \gamma\dfrac{e^{L\tau}-1}{1-\rho e^{L\tau}} - L$ in $[0, -L^{-1}\ln\rho[$. Thus the claim follows once it has been demonstrated that

$$L\lambda_2 < \frac{e^{L\tau}-1}{1-\rho e^{L\tau}} = e^{L\tau} - 1 + (e^{L\tau}-1)\frac{\rho e^{L\tau}}{1-\rho e^{L\tau}} \tag{13}$$

holds for all $L > 0$, $p > 1$, $0 < \rho < 1$ and $0 < \tau < -L^{-1}\ln\rho$, where λ_2 is given by (11). To verify (13), simply note that the latter is implied by

$$\left(\frac{e^{Lp\tau}-1}{p}\right)^{1/p} \left(\frac{e^{Lq\tau}-1}{q}\right)^{1/q} \le e^{L\tau}(e^{L\tau}-1) ,$$

which is (12) with $x = e^{L\tau} > 1$. ■

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References

[1] D. Carnevale, A.R. Teel, D. Nesic, A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems, *IEEE Trans. Automat. Control* **52** (2007), no. 5, 892–897.
[2] D. Nesic, A.R. Teel, Input-output stability properties of networked control systems, *IEEE Trans. Automat. Control* **49** (2004), no. 10, 1650–1667.
[3] G. Walsh, O. Beldiman, L.G. Bushnell, Asymptotic behavior of nonlinear networked control systems, *IEEE Trans. Automat. Control* **46** (2001), no. 7, 1093–1097.
[4] G. Walsh, H. Ye, L.G. Bushnell, Stability analysis of networked control systems, *IEEE Trans. Contr. Syst. Technol.* (2002), vol. 10, 438–446.
[5] M. Tabbara, D. Nesic, A.R. Teel, Stability of wireless and wireline networked control systems, *IEEE Trans. Automat. Control* **52** (2007), no. 9, 1615–1630.
[6] F.H. Clarke, *Optimization and Nonsmooth Analysis*, SIAM, Philadelphia, 1990.