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Global saddle-type dynamics for convex second-order difference equations

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ABSTRACT

A complete, elementary analysis is presented for second-order difference equations $x_n = g(x_{n-1}, x_{n-2})$ where g is strictly monotone and convex in the first quadrant. It is shown that the dynamics of any such equation partitions the phase space into two basins of attraction, one of which is bounded and convex (possibly empty). In the case of two non-empty basins, each point on their common boundary corresponds to an asymptotically 2-periodic solution. The results and examples presented complement previous studies of second-order equations in the literature.

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1. Introduction

Non-linear second-order difference equations exhibit an enormous variety of different dynamical behaviours for which no comprehensive theory appears to be forthcoming [15,27,33]. This is quite unlike the case of second-order *differential* equations which are the subject of a highly developed, classical theory [3,19]. It also contrasts the case of *first*-order difference equations where powerful tools from dynamical systems theory (e.g. coding, renormalization, and transfer operators) can often be applied rather directly [9,10,15]. In the absence of a coherent theory, studies on second-order difference equations naturally focus on specific classes of equations that are of intrinsic mathematical interest, motivated by applications, or both. Although much progress has been achieved for some classes (e.g. rational equations [6,27,32] and the Hénon family [5,10]), many basic questions remain open. This state of affairs is well documented, not least through the monograph [27] and the extensive combined bibliographies of [6,9,15,23,33].

The present article aims to complement the large existing literature (e.g. [1,8,16,17,22, 28,29,37] and references therein) by studying second-order equations

$$x_n = g(x_{n-1}, x_{n-2}), \quad \forall n \geq 3, \tag{1.1}$$

where the non-negative function g, defined in the (closed) first quadrant, belongs to a class of functions that is sufficiently wide to include interesting examples, but at the same time sufficiently narrow to allow strong conclusions to be drawn. Specifically, g will be assumed

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to be strictly *monotone* and *convex*. Monotone difference equations, and to some extent convex or concave equations also, feature a rich theory, together with a well-established range of applications, notably in mathematical biology and economics [21,26,30,34]. While the monotonicity and convexity assumptions individually impose certain restrictions on the dynamics of (1.1), it will become clear herein that only when combined do they force the dynamics to be very simple indeed. Thus the analysis presented here is in the spirit of, and naturally complementary to, the global results of [8,17,18,22,28,29,32].

With all technical details deferred to subsequent sections, to illustrate the main result of this article, fix a real parameter $\alpha \neq -1$, and consider the equation

$$x_n = x_{n-1}^{1+\alpha} + x_{n-2}^{1+\alpha}, \quad \forall n \geq 3, \tag{1.2}$$

which may be thought of as a non-linear analogue of the perennial Fibonacci recursion $x_n = x_{n-1} + x_{n-2}$. Variants of (1.2) with $\alpha < 0$ have been studied extensively; e.g. see [11–13,36] for examples with $\alpha < -1$, and [24–26] for results applicable whenever $-1 < \alpha < 0$. Unlike for these variants (and for the Fibonacci recursion itself), however, $x_n \equiv 0$ is attracting for (1.2) whenever $\alpha > 0$, and there exists another constant solution $x_n \equiv 2^{-1/\alpha}$ which is easily seen to be of saddle-type. For α a positive integer, in particular, (1.2) plays a prominent role in a variety of counting problems that have so far been considered mainly from a number theory point of view [2,18,31]; see also `A000283` and several related entries on the database [38]. In dynamical terms, the presence of an attractor and a saddle naturally raises the question what the global behaviour of (1.2) looks like. This question is answered completely by Theorem 2.2 below which implies the existence of a (unique) convex set $A_0 \supset [0, 2^{-1/\alpha}[^2$ with smooth boundary for which the trichotomy

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } (x_1, x_2) \in A_0, \\ 2^{-1/\alpha} & \text{if } (x_1, x_2) \in \partial A_0, \\ +\infty & \text{if } (x_1, x_2) \notin \overline{A_0}, \end{cases}$$

holds for every solution (x_n) of (1.2); see also Proposition 4.3. Thus A_0 simply is the set of attraction of $x_n \equiv 0$, and ∂A_0 represents the stable manifold of $x_n \equiv 2^{-1/\alpha}$. As the proof of Theorem 2.2 will show, this conclusion hinges on basic structural assumptions regarding the right-hand side of (1.2), but does not depend on its specific form. For instance, an analogous trichotomy holds for

$$x_n = \alpha \cosh(x_{n-1}^2 + 1) \cosh(x_{n-2}^2 + 1), \quad \forall n \geq 3,$$

as well as many other difference equations (1.1); see Proposition 4.5 but cf. also [4].

This article is organized as follows. After introducing and motivating the key assumptions regarding (1.1), Section 2 states the main result and illustrates its assertions by means of a simple example. Section 3 presents an (elementary though somewhat lengthy) proof of the main result, divided into three steps for the reader's convenience. Finally, Section 4 considers two classes of examples in detail, and also highlights several questions that arise naturally from this work.

2. Main theorem – statement and examples

Throughout, denote by $\mathbb{I} = \mathbb{R}^+ \cup \{0\}$ the set of all non-negative real numbers, with the usual topology. As indicated in the Introduction, the purpose of this article is to present a complete analysis of the second-order difference equation

$$x_n = g(x_{n-1}, x_{n-2}), \quad \forall n \geq 3, \tag{1.1}$$

where $(x_1, x_2) \in \mathbb{I}^2$, and $g : \mathbb{I}^2 \to \mathbb{I}$ belongs to a reasonably large family $\mathcal{G}$ of genuinely non-linear functions. To motivate the specific form of $\mathcal{G}$ to be introduced shortly, first consider the first-order analogue of (1.1),

$$x_n = f(x_{n-1}), \quad \forall n \geq 2, \tag{2.1}$$

where $x_1 \in \mathbb{I}$, and the C^1-map $f : \mathbb{I} \to \mathbb{I}$ is assumed to be strictly convex, with $f' \geq 0$. Henceforth, denote by $\mathcal{F}$ the family of all such maps, i.e. let

$$\mathcal{F} = \left\{ f : \mathbb{I} \to \mathbb{I} \text{ is } C^1, \text{strictly convex, and } f'(x) \in \mathbb{I} \ \forall x \in \mathbb{I} \right\}. \tag{2.2}$$

Simple examples of $f \in \mathcal{F}$ include $f(t) = t^{1+\alpha}$ with $\alpha > 0$, and $f(t) = \cosh t$. Clearly, every $f \in \mathcal{F}$ is one-to-one, and a C^1-diffeomorphism of $\mathbb{R}^+$. Note that if $f_1, f_2 \in \mathcal{F}$ then $\alpha_1 f_1 + \alpha_2 f_2 \in \mathcal{F}$ and $f_1(\alpha_1 \,\cdot\,) \in \mathcal{F}$ for all $\alpha_1, \alpha_2 > 0$, but also $f_1 f_2 \in \mathcal{F}$ as well as $f_1 \circ f_2 \in \mathcal{F}$. Thus for instance $f = \alpha \cosh(\,\cdot\,^2 + 1) \in \mathcal{F}$ for all $\alpha > 0$.

To conveniently describe the asymptotic behaviour of (2.1) for arbitrary $f : \mathbb{I} \to \mathbb{I}$, denote by A_ξ the *set of attraction* of any $\xi \in \mathbb{I} \cup \{+\infty\}$ under (2.1), that is,

$$A_\xi = \left\{ x_1 \in \mathbb{I} : \lim_{n\to\infty} x_n = \xi \right\}.$$

The (possibly empty) set A_ξ is f-invariant, i.e. $f^{-1}(A_\xi) = A_\xi$, and $A_\xi \cap A_\eta = \varnothing$ whenever $\xi \neq \eta$. For $\xi \in \mathbb{I}$, clearly $f(\xi) = \xi$ implies that $A_\xi \neq \varnothing$, and the converse is also true, provided that f is continuous at ξ. It is well known that, depending on the specific properties of f, the sets A_ξ may be complicated whenever non-empty, and so may be $\mathbb{I} \setminus \bigcup_\xi A_\xi$; e.g. see [9,10,15]. It is easy to check, however, that none of this complexity can occur whenever $f \in \mathcal{F}$; see also Figure 1.

Proposition 2.1: *For each $f \in \mathcal{F}$ precisely one of the following alternatives applies:*

(i) *There exists a unique $\xi \in \mathbb{I} \cup \{+\infty\}$ such that $A_\xi = \mathbb{I}$;*
(ii) *There exists a unique $\xi \in \mathbb{I}$ such that A_ξ is a non-empty bounded interval containing 0, and $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}$. Moreover, if $x_1 \in \partial A_\xi$ then $f(x_1) = x_1$.*

Informally put, Proposition 2.1 asserts that for $f \in \mathcal{F}$ the entire phase space $\mathbb{I}$ is the disjoint union of at most two sets (in fact, intervals) $\overline{A_\xi}$ and $A_{+\infty}$, the former being a bounded interval containing 0 that can only have a fixed point of f as its boundary. Note that in order to reach this conclusion, the properties defining $\mathcal{F}$ in (2.2) are, in a sense, minimal: On the one hand, a convex map $f : \mathbb{I} \to \mathbb{I}$ with $f' \geq 0$ yet failing to be strictly convex may have an interval of fixed points, in which case $A_\xi \neq \varnothing$ for uncountably many ξ. On the other hand, it is well known that a strictly convex map f can generate *much* more complicated behaviour of (2.1) when $-f'(0) > 0$ is sufficiently large and f has two repelling fixed points [9,10,15].

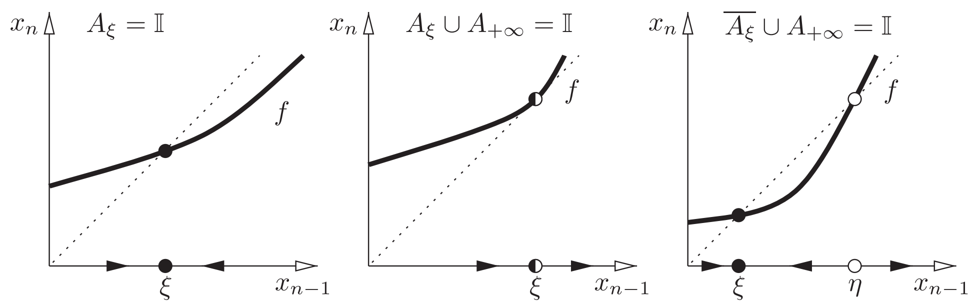

Figure 1. Given $f \in \mathcal{F}$, either $A_\xi = \mathbb{I}$ for a unique ξ (left), or else $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}$, and the bounded interval A_ξ may or may not contain the fixed point at its right end; see Proposition 2.1.

The main result of this article, Theorem 2.2 below, asserts that the clear-cut alternatives appearing in Proposition 2.1 to a large extent persist for (1.1) and the phase space $\mathbb{I}^2$ — provided that g belongs to a two-dimensional analogue of (2.2). Concretely, it will be assumed that the C^1-function $g : \mathbb{I}^2 \to \mathbb{I}$ is strictly convex, with $g_{x_1}(0, x_2) \geq 0$, $g_{x_2}(x_1, 0) \geq 0$ for every $x \in \mathbb{I}^2$. Throughout, denote by $\mathcal{G}$ the family of all such functions, that is,

$$\mathcal{G} = \left\{ g : \mathbb{I}^2 \to \mathbb{I} \text{ is } C^1, \text{strictly convex, and } \nabla g(x) \in \mathbb{I}^2 \ \ \forall x \in \mathbb{I}^2 \right\}. \tag{2.3}$$

Note that if $g \in \mathcal{G}$ then $g(\,\cdot\,, x_2), g(x_1, \,\cdot\,) \in \mathcal{F}$ for every $x \in \mathbb{I}^2$. The family $\mathcal{G}$ is reasonably large. For instance, if $f_1, f_2 \in \mathcal{F}$ then $f_1 \oplus f_2 \in \mathcal{G}$, where $f_1 \oplus f_2(x) = f_1(x_1) + f_2(x_2)$ for all $x \in \mathbb{I}^2$.

In analogy to the first-order case, given any $g : \mathbb{I}^2 \to \mathbb{I}$ and $\xi \in \mathbb{I} \cup \{+\infty\}$, consider again the set of attraction of ξ under (1.1), i.e. let

$$A_\xi = \left\{ (x_1, x_2) \in \mathbb{I}^2 : \lim_{n\to\infty} x_n = \xi \right\}.$$

Usage of the same symbol A_ξ in connection with both (1.1) and (2.1) should not cause confusion, since it will always be clear from the context whether $A_\xi \subset \mathbb{I}^2$ or $A_\xi \subset \mathbb{I}$, respectively. As in the one-dimensional setting, but perhaps more dramatically so, the sets A_ξ, as well as $\mathbb{I}^2 \setminus \bigcup_\xi A_\xi$ may be very complicated, even for (convex) functions as innocent-looking as $g(x) = a_1 + x_1^2 + a_2 x_2$ with $a_1, a_2 \in \mathbb{R}$; see e.g. [5,10]. However, the following analogue of Proposition 2.1 completely rules out such complexity whenever $g \in \mathcal{G}$, and instead guarantees a neat partition of the phase space $\mathbb{I}^2$. (Here and throughout, all topological terms are understood relative to the product topology of $\mathbb{I}^2$.) In some sense, the result may be considered a counterpart of known results for *concave* systems, as established e.g. in [26].

Theorem 2.2: *For each $g \in \mathcal{G}$ precisely one of the following alternatives applies:*

(i) *There exists a unique $\xi \in \mathbb{I} \cup \{+\infty\}$ such that $A_\xi = \mathbb{I}^2$, i.e. $\lim_{n\to\infty} x_n = \xi$ for every solution (x_n) of (1.1);*

(ii) *There exists a unique $\xi \in \mathbb{I}$ such that A_ξ is a non-empty bounded convex set containing $(0, 0)$, and $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}^2$. Moreover, if $(x_1, x_2) \in \partial A_\xi$ then (x_n) is asymptotically 2-periodic, i.e. the sequences (x_{2n-1}) and (x_{2n}) generated by (1.1) both converge.*

The following example illustrates the alternatives in Theorem 2.2; a more detailed analysis is presented in Example 4.1.

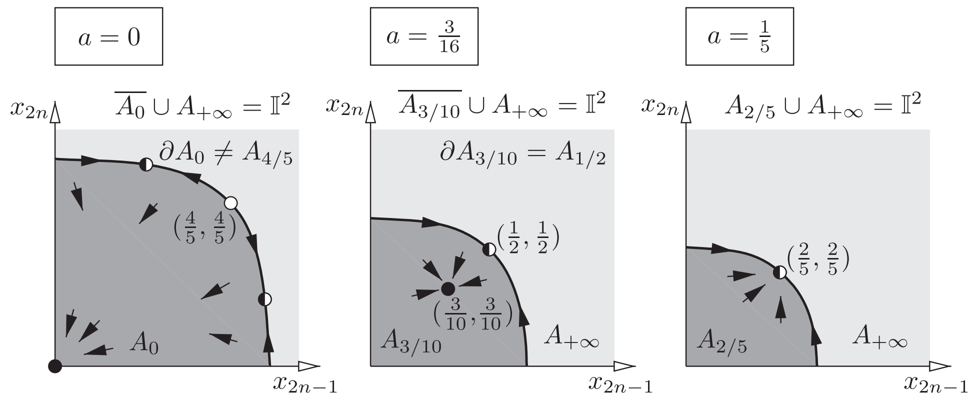

Figure 2. For $g(x) = a + \frac{1}{4}x_1^2 + x_2^2$, alternative (ii) of Theorem 2.2 applies whenever $0 \leq a \leq \frac{1}{5}$, but the compact convex set $\overline{A_\xi}$, as well as the dynamics of (2.4) within it, vary with a; see Examples 2.3 and 4.1.

Example 2.3: For any $a \geq 0$, consider the difference equation

$$x_n = a + \tfrac{1}{4}x_{n-1}^2 + x_{n-2}^2, \quad \forall n \geq 3, \tag{2.4}$$

i.e. let $g(x) = a + \frac{1}{4}x_1^2 + x_2^2$. Plainly, $g \in \mathcal{G}$, so Theorem 2.2 applies, and shows that the dynamics of (2.4) always is very simple, with the details depending on the value of a as follows; see also Figure 2.

(i) If $a > \frac{1}{5}$ then $A_{+\infty} = \mathbb{I}^2$.

(ii) If $a = \frac{1}{5}$ then $A_{2/5} \cup A_{+\infty} = \mathbb{I}^2$ where $A_{2/5}$ is a compact convex neighbourhood of $(0, 0)$ with C^∞-boundary. Note that although it resembles a quarter-disc, the set $A_{2/5}$ is *not* symmetric w.r.t. the line $x_1 = x_2$.

(iii) If $a < \frac{1}{5}$ then $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}^2$, where $\xi = \frac{2}{5}\left(1 - \sqrt{1 - 5a}\right)$, and A_ξ is open and convex, again resembling a quarter-disc. The precise nature of points (x_1, x_2) in ∂A_ξ depends on the value of a.
If $a \geq \frac{1}{9}$ then simply $\partial A_\xi = A_\eta$, with $\eta = \frac{4}{5} - \xi > \xi$. In other words, $\lim_{n\to\infty} x_n = \eta$ whenever $(x_1, x_2) \in \partial A_\xi$. For instance, for $a = \frac{3}{16}$ one finds $\xi = \frac{3}{10}$, $\eta = \frac{1}{2}$, hence $\overline{A_{3/10}} \cup A_{+\infty} = \mathbb{I}^2$ and $\partial A_{3/10} = A_{1/2}$.
If, however, $a < \frac{1}{9}$ then every $(x_1, x_2) \in \partial A_\xi$ different from (η, η) yields an asymptotically 2-periodic solution. For instance, for $a = 0$ one finds $\xi = 0$, $\eta = \frac{4}{5}$, and for every solution (x_n) of (2.4) with $(x_1, x_2) \in \partial A_0$ and $x_1 \neq \frac{4}{5}$,

$$\lim\nolimits_{n\to\infty} x_{2n-1} = \tfrac{2}{15}\left(5 + \sqrt{5}\right) \quad \text{and} \quad \lim\nolimits_{n\to\infty} x_{2n} = \tfrac{2}{15}\left(5 - \sqrt{5}\right),$$

or vice versa, depending on whether $x_1 > \frac{4}{5}$ or $x_1 < \frac{4}{5}$.

3. Main theorem – proof

This section provides a *Proof of Theorem* 2.2. For the reader's convenience, the argument is divided into three main steps; throughout, let $g \in \mathcal{G}$ be a given function.

Step I: monotone maps G and f_g associated with (1.1)

Denote by $\preceq$ the standard partial order on $\mathbb{R}^2$, that is, $x \preceq y$ for $x, y \in \mathbb{R}^2$ if and only if $y - x \in \mathbb{I}^2$. As usual, write $x \prec y$ whenever $x \preceq y$ yet $x \neq y$, and write $x \ll y$ if both $x_1 < y_1$ and $x_2 < y_2$. Denote $(1, 1) \in \mathbb{I}^2$ simply by $\mathbf{1}$, and observe that $\min\{x_1, x_2\}\,\mathbf{1} \preceq x \preceq \max\{x_1, x_2\}\,\mathbf{1}$ for every $x \in \mathbb{I}^2$.

For any $g \in \mathcal{G}$, rewrite (1.1) in the form

$$(x_{n-1}, x_n) = \bigl(x_{n-1}, g(x_{n-1}, x_{n-2})\bigr) = G(x_{n-2}, x_{n-1})\,, \quad \forall n \geq 3\,,$$

with the map $G : \mathbb{I}^2 \to \mathbb{I}^2$ given by $G(x) = \bigl(x_2, g(x_2, x_1)\bigr)$ for all $x \in \mathbb{I}^2$. Note that G is a homeomorphism of $\mathbb{I}^2$, and a diffeomorphism of $(\mathbb{R}^+)^2$. If $x \preceq y$ then

$$G(y) - G(x) = \bigl(y_2 - x_2, g(y_2, y_1) - g(x_2, x_1)\bigr) \succeq 0\,\mathbf{1}\,,$$

from which it is clear that $G(x) \prec G(y)$ whenever $x \prec y$. Thus G is *strictly order-preserving* [20, p. 9]. Moreover, if $x \prec y$ then, with $G^2 := G \circ G$,

$$G^2(y) - G^2(x) = \Bigl(g(y_2, y_1) - g(x_2, x_1), g\bigl(g(y_2, y_1), y_2\bigr) - g\bigl(g(x_2, x_1), x_2\bigr)\Bigr) \gg 0\,\mathbf{1}\,,$$

and so, unlike G, the homeomorphism G^2 has the even stronger property of being *strongly order-preserving*, that is, $G^2(x) \ll G^2(y)$ whenever $x \prec y$. Though strict and strong order-preservation are important properties, recall that they per se do not preclude non-trivial long-time behaviour of difference equations, in stark contrast to their continuous-time counterparts [20,21,34,35]

Next, associate with $g \in \mathcal{G}$ the map $f_g : \mathbb{I} \to \mathbb{I}$, where $f_g(t) = g(t, t)$ for all $t \in \mathbb{I}$. Note that $f_g \in \mathcal{F}$, and hence Proposition 2.1 applies. In particular, the set $\mathbb{J} := \{t \geq 0 : f_g(t) \leq t\}$ is a closed subinterval of $\mathbb{I}$, and so, with the appropriate $\xi, \eta \in \mathbb{I}$ with $\xi \leq \eta$,

$$\mathbb{J} = \varnothing\,, \quad \text{or} \quad \mathbb{J} = [\xi, +\infty[\,, \quad \text{or} \quad \mathbb{J} = [\xi, \eta]\,. \tag{3.1}$$

The first two cases in (3.1) are easily analyzed, via a standard argument (see e.g. [7,20,25]) that is recalled here briefly for the reader's convenience. For instance, if $\mathbb{J} = [\xi, +\infty[$ with $\xi \in \mathbb{I}$ then $f_g(t) > t$ for all $t < \xi$, and $f_g(t) < t$ for all $t > \xi$. Given $x \in \mathbb{I}^2$, let $\xi^- = \min\{x_1, x_2, \xi\}$ and $\xi^+ = \max\{x_1, x_2, \xi\}$, and observe that

$$f_g^{\lfloor n/2 \rfloor}(\xi^-)\,\mathbf{1} \preceq G^n(x) \preceq f_g^{\lfloor n/2 \rfloor}(\xi^+)\,\mathbf{1}\,, \quad \forall n \geq 1\,; \tag{3.2}$$

here, $\lfloor s \rfloor$ denotes the largest integer not larger than $s \in \mathbb{R}$. Since $\lim_{n\to\infty} f_g^n(t) = \xi$ for all $t \geq 0$, it follows from (3.2) that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} G^{n-1}(x_1, x_2)_1 = \xi$ for every solution (x_n) of (1.1), i.e. $A_\xi = \mathbb{I}^2$. The case of $\mathbb{J} = \varnothing$ can be dealt with in a completely analogous manner, yielding $A_{+\infty} = \mathbb{I}^2$. Thus the first two cases in (3.1) both correspond to alternative (i) of Theorem 2.2. In Steps II and III below it will be shown that the third case corresponds to alternative (ii) of that theorem.

Step II: existence of invariant graphs

Consider the remaining case $\mathbb{J} = [\xi, \eta]$ in (3.1), with $0 \le \xi \le \eta$. If $\eta = 0$ then $f_g(t) > t$ for all $t > 0$. In this case, for every $x \in \mathbb{I}^2 \setminus \{(0,0)\}$ clearly $G^2(x) \gg t\,\mathbf{1}$ with the appropriate $t > 0$, and consequently $G^{n+2}(x) \gg f_g^{\lfloor n/2 \rfloor}(t)\,\mathbf{1}$ for every $n \ge 0$. This shows that $A_{+\infty} = \mathbb{I}^2 \setminus \{(0,0)\}$ and $A_0 = \{(0,0)\}$. Quite trivially, therefore, all assertions of Theorem 2.2(ii) are correct whenever $\eta = 0$.

For the remainder of the proof, assume that $\mathbb{J} = [\xi, \eta]$ with $\eta > 0$, and define three associated 'rectangles' in $\mathbb{I}^2$,

$$B = [0, \eta]^2, \quad C_\ell = [0, \eta] \times]\eta, +\infty[, \quad C_r =]\eta, +\infty[\times [0, \eta].$$

Note that $B = \{x \in \mathbb{I}^2 : x \preceq \eta\,\mathbf{1}\}$. The rectangles B, C_r, C_ℓ are disjoint and, together with the 'infinite square'

$$\mathbb{I}^2 \setminus (B \cup C_\ell \cup C_r) =]\eta, +\infty[^2 = \{x \in \mathbb{I}^2 : x \gg \eta\,\mathbf{1}\},$$

they form a partition of $\mathbb{I}^2$; see also Figure 3. Recall that ξ and η are fixed by f_g, and hence $\xi\,\mathbf{1}$ and $\eta\,\mathbf{1}$ are fixed points of G. If $x \in B$ then $G(x) \preceq G(\eta\,\mathbf{1}) = \eta\,\mathbf{1}$, and so $G^{-1}(B) \supset B$. Similarly, $G^{-1}(B \cup C_\ell \cup C_r) \subset B \cup C_\ell \cup C_r$. With $B_n^- := G^{-n}(B)$ and $B_n^+ := G^{-n}(B \cup C_\ell \cup C_r)$ for every $n \ge 0$, therefore, $B = B_0^- \subset B_1^- \subset B_2^- \subset \cdots$ and $B \cup C_\ell \cup C_r = B_0^+ \supset B_1^+ \supset B_2^+ \supset \cdots$, but also $B_m^- \subset B_n^+$ for all m, n. Note that the set B_2^+ is bounded. Moreover, observe that $G(x) \in C_r$ requires $x_2 > \eta$ as well as $g(x_2, x_1) \le \eta$, hence $x_1 < \eta$, and so $x \in C_\ell$. In other words, $G^{-1}(C_r) \subset C_\ell$, and analogously $G^{-1}(C_\ell) \subset C_r$.

Assume now that, for some $n \ge 0$,

$$B_n^- \text{ is convex, and if } x \in B_n^- \text{ then } \{y \in \mathbb{I}^2 : y \preceq x\} \subset B_n^-. \tag{3.3}$$

Given any $x, \widetilde{x} \in B_{n+1}^-$ and $0 \le t \le 1$, it follows from (3.3) and the convexity of g that $G\big((1-t)x + t\widetilde{x}\big) \preceq (1-t)G(x) + tG(\widetilde{x}) \in B_n^-$, and so $(1-t)x + t\widetilde{x} \in B_{n+1}^-$, i.e. B_{n+1}^- is convex. Also, if $y \preceq x \in B_{n+1}^-$ then $G(y) \preceq G(x) \in B_n^-$, hence $y \in B_{n+1}^-$, by (3.3). The latter thus holds with n replaced by $n+1$, and in fact for all $n \ge 0$, since trivially it holds for $n = 0$. In particular, for every n the set $B_n^- \supset B$ is bounded and convex, contains the point $\eta\,\mathbf{1}$ but is disjoint from $\mathbb{I}^2 \setminus (B \cup C_\ell \cup C_r)$. Clearly, the set $B_n^- \cap (B \cup C_\ell)$ has all these properties as well. In fact, since $x \notin B_2^+$ whenever $x \succ \eta\,\mathbf{1}$, except for the point $\eta\,\mathbf{1}$ the set B_n^-, and also $B_n^- \cap (B \cup C_\ell)$, is contained in the interior of $B \cup C_\ell \cup C_r$. It is possible, therefore, to write $B_n^- \cap (B \cup C_\ell)$ as

$$B_n^- \cap (B \cup C_\ell) = \left\{x \in \mathbb{I}^2 : 0 \le x_1 \le \eta, 0 \le x_2 \le \varphi_n^-(x_1)\right\},$$

with a unique function $\varphi_n^- : [0, \eta] \to \mathbb{R}$ that is continuous, concave, non-increasing, with $\varphi_n^-(\eta) = \eta$. Since $B_n^- \subset B_{n+1}^-$, the sequence $\big(\varphi_n^-(s)\big)$ is non-decreasing for every $0 \le s \le \eta$. Recall that B_2^+ is bounded, and $B_n^- \subset B_2^+$, so each function φ_n is bounded above, independently of n. It follows that $\varphi^-(s) := \lim_{n\to\infty} \varphi_n^-(s)$ defines a concave, non-increasing function on $[0, \eta]$, with $\varphi^-(\eta) = \eta$. Clearly, φ^- is continuous, and hence $\varphi_n^- \to \varphi^-$ uniformly on $[0, \eta]$, by Dini's Theorem [14]. Picking any point $x = \big(s, \varphi_{n+2}^-(s)\big) \in \partial B_{n+2}^- \cap (B \cup C_\ell)$, recall that, by construction,

$$G^2(x) = \Big(g(\varphi_{n+2}^-(s), s), g\big(g(\varphi_{n+2}^-(s), s), \varphi_{n+2}^-(s)\big)\Big),$$

and consequently, for all $n \geq 0$,

$$\varphi_n^-\big(g(\varphi_{n+2}^-(s), s)\big) = g\big(g(\varphi_{n+2}^-(s), s), \varphi_{n+2}^-(s)\big), \quad \forall s \in [0, \eta]. \tag{3.4}$$

Letting $n \to \infty$ in (3.4), it follows from the uniform convergence $\varphi_n^- \to \varphi^-$ that

$$\varphi^-\big(g(\varphi^-(s), s)\big) = g\big(g(\varphi^-(s), s), \varphi^-(s)\big), \quad \forall s \in [0, \eta],$$

which simply says that graph $\varphi^- := \big\{\big(s, \varphi^-(s)\big) : s \in [0, \eta]\big\} \subset \mathbb{I}^2$ is forward invariant under G^2. When restricted to graph φ^-, the map G^2 induces the continuous map $h^- : [0, \eta] \to [0, \eta]$ given by $h^-(s) = g(\varphi^-(s), s)$, that is,

$$G^2\big(s, \varphi^-(s)\big) = \big(h^-(s), \varphi^- \circ h^-(s)\big), \quad \forall s \in [0, \eta].$$

Note that $h^-(\eta) = \eta$, and since G^2 is one-to-one, so is h^-. Thus h^- is increasing.

A completely analogous analysis confirms that, for every $n \geq 1$,

$$B_n^+ \cap (B \cup C_\ell) = \big\{x \in \mathbb{I}^2 : 0 \leq x_1 \leq \eta, 0 \leq x_2 \leq \varphi_n^+(x_1)\big\},$$

with a unique continuous, concave and non-increasing function $\varphi_n^+ : [0, \eta] \to \mathbb{R}$ satisfying $\varphi_n^+(\eta) = \eta$. Since $B_n^+ \supset B_{n+1}^+$, the sequence $\big(\varphi_n^+(s)\big)$ is non-increasing, and just as before, $\varphi_n^+(s) \to \lim_{n\to\infty} \varphi_n^+(s) =: \varphi^+(s)$ uniformly on $[0, \eta]$. From $B_m^- \subset B_n^+$ it is clear that $\varphi_m^- \leq \varphi_n^+$ for all m, n, and consequently $\varphi^- \leq \varphi^+$. Again, graph φ^+ is forward invariant under G^2, and the latter induces an increasing continuous map $h^+ : [0, \eta] \to [0, \eta]$ with $h^+(\eta) = \eta$; explicitly, $h^+ = g(\varphi^+(\,\cdot\,), \cdot)$, and so $h^+ \geq h^-$.

To relate the above considerations to the asymptotic behaviour of (1.1), note on the one hand that if $x \in B \cup C_\ell$ and $x_2 < \varphi^-(x_1)$ then $G^{2N}(x) \in B \setminus \{\eta\,\mathbf{1}\}$ for some integer $N \geq 1$, hence $s^-\,\mathbf{1} \preceq G^{2N+2}(x) \preceq s^+\,\mathbf{1}$ for some $0 < s^- < s^+ < \eta$, and

$$f_g^{\lfloor n/2 \rfloor}(s^-)\,\mathbf{1} \preceq G^{2N+2+n}(x) \preceq f_g^{\lfloor n/2 \rfloor}(s^+)\,\mathbf{1}, \quad \forall n \geq 0,$$

from which it is clear that $\lim_{n\to\infty} x_n = \xi$, i.e. $x \in A_\xi$. If, on the other hand, $x \in B \cup C_\ell$ but $x_2 > \varphi^+(x_1)$ then $G^{2N}(x) \in \mathbb{I}^2 \setminus (B \cup C_\ell \cup C_r)$ for some $N \geq 1$, which implies $\lim_{n\to\infty} x_n = +\infty$, i.e. $x \in A_{+\infty}$. A similar analysis applies to the sets $G^{-1}(\text{graph}\,\varphi^\pm) \subset C_r$ which can be written as

$$G^{-1}(\text{graph}\,\varphi^\pm) = \big\{x \in \mathbb{I}^2 : 0 \leq x_2 \leq \eta, 0 \leq x_1 \leq \psi^\pm(x_2)\big\}$$

with continuous, concave and non-increasing functions $\psi^\pm$. (Explicitly, $\psi^\pm(s)$ is given by $g(s, \cdot)^{-1} \circ \varphi^\pm(s)$ for all $0 \leq s \leq \eta$; here and below, expressions containing the symbol $\pm$ or $\mp$ are to be read as two separate expressions containing only the upper and only the lower signs, respectively.) If $x \in B \cup C_r$ and $x_1 < \psi^-(x_2)$ then $x \in A_\xi$, whereas if $x \in B \cup C_r$ and $x_1 > \psi^+(x_2)$ then $x \in A_{+\infty}$. From this, it is evident that the only points $(x_1, x_2) \in \mathbb{I}^2$ whose ultimate fate under (1.1) is as yet undecided are precisely the ones with

$$0 \leq x_1 < \eta, \varphi^-(x_1) \leq x_2 \leq \varphi^+(x_2) \quad \text{or} \quad 0 \leq x_2 < \eta, \psi^-(x_2) \leq x_1 \leq \psi^+(x_2); \tag{3.5}$$

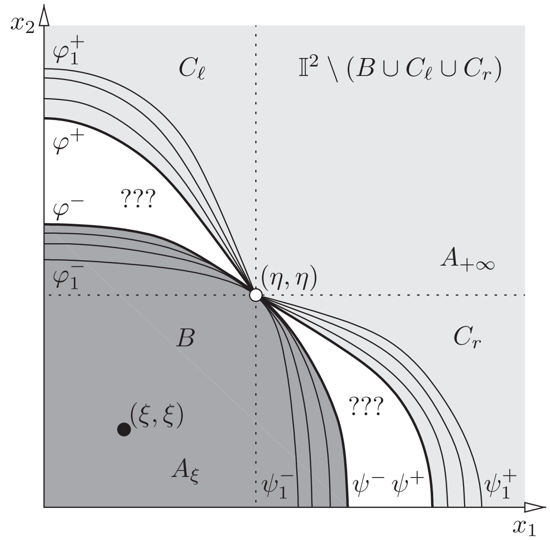

Figure 3. Illustrating Step II in the proof of Theorem 2.2 (schematic); the white areas marked ??? indicate the set of points specified by (3.5).

see also Figure 3. Step III below will demonstrate that the assumption $g \in \mathcal{G}$ entails $\varphi^- = \varphi^+$ (and hence also $\psi^- = \psi^+$), which in turn implies that in fact the set of points specified by (3.5) is but a single curve.

Step III: detailed dynamical analysis

With the functions $\varphi^\pm$ constructed in Step II, consider the set

$$U := \{0 \le s \le \eta : \varphi^-(s) < \varphi^+(s)\} \subset [0, \eta[\ .$$

If $s \in U$ then

$$\begin{aligned}\varphi^+ \circ h^-(s) = \varphi^+\big(g(\varphi^-(s), s)\big) &\ge \varphi^+\big(g(\varphi^+(s), s)\big) \\ &= g\big(h^+(s), \varphi^+(s)\big) > g\big(h^-(s), \varphi^-(s)\big) = \varphi^- \circ h^-(s)\,,\end{aligned}$$

which in turn shows that $h^-(s) \in U$, i.e. $h^-(U) \subset U$. Conversely, if $s \notin U$ then $h^-(s) = h^+(s)$, and hence

$$\big(h^-(s), \varphi^- \circ h^-(s)\big) = G^2\big(s, \varphi^\pm(s)\big) = \big(h^-(s), \varphi^+ \circ h^-(s)\big)\,,$$

showing that $h^-(s) \notin U$, i.e. $(h^-)^{-1}(U) \subset U$. In summary, the set U is invariant under h^-. Similarly, U is invariant under h^+.

As all assertions in Theorem 2.2(ii) will follow easily afterwards, the main goal of this step is to show that in fact $U = \varnothing$. For a preparatory consideration towards this goal, assume $h^-(s) = s$ for some $0 \le s < \eta$. Then $x = \big(s, \varphi^-(s)\big) \in \mathbb{I}^2$ is a fixed point of G^2. By the construction of φ^-, every point $y \prec x$ is contained in A_ξ, and hence the derivative $DG^2|_x$ cannot have two eigenvalues with modulus less than one. Observe that

$$DG^2|_x = \begin{bmatrix} g_{x_2}\circ G(x) & g_{x_1}\circ G(x) \\ g_{x_1}(x)g_{x_2}\circ G(x) & g_{x_1}(x)g_{x_1}\circ G(x) + g_{x_2}(x) \end{bmatrix} =: \begin{bmatrix} \alpha_4 & \alpha_3 \\ \alpha_1\alpha_4 & \alpha_1\alpha_3 + \alpha_2 \end{bmatrix} \quad (3.6)$$

is a positive matrix, with eigenvalues $\lambda_1 > \lambda_2 > 0$, and as stated above, $\lambda_1 \geq 1$. The eigenspace corresponding to λ_1 is spanned by some $z = (z_1, z_2)$ with $z_1, z_2 > 0$. Next, observe that, for every $t > 0$,

$$\begin{aligned} G^2(x+tz) &= \big(g(\varphi^-(s)+tz_2, s+tz_1), g(g(\varphi^-(s)+tz_2, s+tz_1), \varphi^-(s)+tz_2)\big) \\ &\gg \big(s + t\alpha_4 z_1 + t\alpha_3 z_2, g(s+t\alpha_4 z_1 + t\alpha_3 z_2, \varphi^-(s)+tz_2)\big) \\ &\succeq (s + t\alpha_4 z_1 + t\alpha_3 z_2, \varphi^-(s) + t\alpha_1\alpha_4 z_1 + t(\alpha_1\alpha_3+\alpha_2)z_2) \;=\; x + t\lambda_1^2 z\,, \end{aligned}$$

where both inequalities are due to the strict convexity of g. Suppose now that $\varphi^-(s) < \varphi^+(s)$. Since $t \mapsto \varphi^-(s) + tz_2$ is increasing while $t \mapsto \varphi^+(s+tz_1)$ is non-increasing (and both functions are continuous), there exists $0 < t < (\eta - s)z_1^{-1}$ with $\varphi^-(s) + tz_2 = \varphi^+(s+tz_1)$. But then

$$\begin{aligned} \big(h^+(s+tz_1), \varphi^+\circ h^+(s+tz_1)\big) = G^2\big(s+tz_1, \varphi^+(s+tz_1)\big) = G^2(x+tz) \\ \gg x + t\lambda_1^2 z \succeq \big(s+tz_1, \varphi^+(s+tz_1)\big)\,, \end{aligned}$$

which is impossible since φ^+ is non-increasing. In summary, if $x = \big(s, \varphi^-(s)\big)$ is a fixed point of G^2 then necessarily $\varphi^-(s) = \varphi^+(s)$.

Using this preparatory step, it will now be shown that indeed $U = \varnothing$, as claimed earlier. Suppose by way of contradiction that $s \in U \neq \varnothing$. As seen above, the point $x = \big(s, \varphi^-(s)\big)$ cannot be fixed by G^2, and hence either $h^-(s) > s$ or $h^-(s) < s$.

Assume first that $h^-(s) > s$, and let $V = \{s \leq t \leq \eta : h^-(t) \leq t\}$. Note that V is compact, $\eta \in V$, and $s < v := \min V$. By the monotonicity and continuity of h^-, $\lim_{n\to\infty} (h^-)^n(s) = v$ and $h^-(v) = v$. Thus $y := \big(v, \varphi^-(v)\big)$ is fixed by G^2, and consequently $\varphi^-(v) = \varphi^+(v)$ as well as $h^+(v) = v$. Since $\big(s, \varphi^-(s)\big) \prec \big(s, \varphi^+(s)\big)$,

$$\begin{aligned} \big((h^-)^n(s), \varphi^-\circ(h^-)^n(s)\big) = G^{2n}\big(s, \varphi^-(s)\big) \lll G^{2n}\big(s, \varphi^+(s)\big) \\ = \big((h^+)^n(s), \varphi^+\circ(h^+)^n(s)\big)\,, \end{aligned}$$

and so $(h^-)^n(s) < (h^+)^n(s) \leq (h^+)^n(v) = v$, which shows that $\lim_{n\to\infty} (h^+)^n(s) = v$ also. Recall now that the fixed point y of the diffeomorphism G^2 cannot be a sink (because $G^{2n}(z) \to \xi\,\mathbf{1}$ for some points z arbitrarily close to y) but, as just seen, has at least one (possibly only weakly) attracting direction. In terms of the two eigenvalues $\lambda_{1,2}$ of $DG^2|_z$ this means that necessarily $\lambda_1 \geq 1$ and $0 < \lambda_2 \leq 1$. (Note that $\alpha_2 > 0$ in (3.6), and so it is impossible to have $\lambda_1 = \lambda_2 = 1$.) If $\lambda_2 < 1$ then y has a unique one-dimensional local stable manifold whereas if $\lambda_2 = 1$ (and hence $\lambda_1 > 1$) then y has a (not necessarily unique) one-dimensional local centre manifold. In either case, by (a hyperbolic or a non-hyperbolic version, respectively, of) the Hartman–Grobman Theorem [19], it is clear that, with the appropriate $\varepsilon > 0$, $\varphi^-(t) = \varphi^+(t)$ for all $v - \varepsilon \leq t \leq v$. But since $v - \varepsilon < (h^\pm)^n(s) < v$ for all sufficiently large n, clearly $(h^\pm)^n(s) \notin U$, that is, $s \notin (h^\pm)^{-n}(U) = U$, contradicting the initial assumption $s \in U$. Thus, if $s \in U$ then $h^-(s) > s$ is impossible.

Assume in turn that $s \in U$ but $h^-(s) < s$, and similarly to before, consider the set $W := \{0 \leq t \leq s : h^-(t) \geq t\}$. Notice that $W \neq \varnothing$, as $0 \in W$, and the argument is now analogous to the case of $h^-(s) > s$ considered earlier: Let $w = \max W$. Then $\lim_{n\to\infty}(h^-)^n(s) = w$ and $h^-(w) = w$. Since $\big(w, h^-(w)\big)$ is fixed by G^2, $\varphi^-(w) = \varphi^+(w)$, and it is readily deduced from the strict order preservation of G^2 that $\lim_{n\to\infty}(h^+)^n(s) = w$ as well. The fixed point $\big(w, h^-(w)\big)$ of G^2 is not a sink, which, in complete analogy to before, implies that $\varphi^-(t) = \varphi^+(t)$ for all $w \le t \le w + \varepsilon$ with the appropriate $\varepsilon > 0$. Since $w \le (h^\pm)^n(s) < w + \varepsilon$ for all sufficiently large n, it follows that $(h^\pm)^n(s) \notin U$, hence $s \notin (h^\pm)^{-n}(U) = U$, again contradicting the initial assumption $s \in U$.

In summary, it has now been proved that if $\mathbb{J} = [\xi, \eta]$ with $\eta > 0$ then the functions φ^- and φ^+ constructed in Step II coincide, and so do ψ^- and ψ^+. From this, the assertions of Theorem 2.2(ii) are deduced easily: First recall from Step II that $\overline{\bigcup_{n\ge0} B_n^-} \subset \overline{A_\xi \cup \{\eta\mathbf{1}\}}$ and $\mathbb{I}^2 \setminus A_{+\infty} \subset \bigcap_{n\ge0} B_n^+$. Since $\varphi^- = \varphi^+$ and $\psi^- = \psi^+$, it follows that $\overline{\bigcup_{n\ge0} B_n^-} = \bigcap_{n\ge0} B_n^+ = \overline{A_\xi}$. Hence $\overline{A_\xi}$ is a compact convex set, and $\partial\overline{A_\xi} = \partial A_\xi$ consists precisely of the points $x \in \mathbb{I}^2$ specified by (3.5). In particular, A_ξ is non-empty, bounded, and convex, with $(0,0) \in A_\xi$. Moreover, if $(x_1, x_2) \in \partial A_\xi$ then, as also seen in Step II, the sequences given by

$$x_{2n-1} = (G^2)^{n-1}(x_1, x_2)_1 \quad \text{and} \quad x_{2n} = (G^2)^{n-1}(x_1, x_2)_2\,, \quad \forall n \ge 1\,,$$

converge (to the first and second component, respectively, of a fixed point of G^2), due to the continuity and monotonicity of $h^- = h^+$. In other words, the solution (x_n) of (1.1) is asymptotically 2-periodic whenever $(x_1, x_2) \in \partial A_\xi$. $\square$

Remark 3.1: Since the functions $\varphi^- = \varphi^+$ and $\psi^- = \psi^+$ are concave, the curve $\partial A_\xi = \operatorname{graph}\varphi^- \cup \operatorname{graph}\psi^-$ is Lipschitz continuous. From Step III above, it is evident that ∂A_ξ is a C^1-submanifold of $\mathbb{I}^2$ which in fact may be as smooth as g.

4. Concluding examples and remarks

This final section illustrates Theorem 2.2 in the context of two specific families of recursions. It also hints at possible follow-up questions for subsequent studies.

Example 4.1: Given real numbers $a \ge 0$ and $b_1, b_2 > 0$, consider the recursion

$$x_n = a + b_1 x_{n-1}^2 + b_2 x_{n-2}^2\,, \quad \forall n \ge 3\,, \tag{4.1}$$

thus $g(x) = a + b_1 x_1^2 + b_2 x_2^2$. Clearly, $g \in \mathcal{G}$, and (4.1) contains (2.4) as the special case $b_1 = \frac{1}{4}$, $b_2 = 1$. For convenience, let $a^* = \frac{1}{4}(b_1 + b_2)^{-1}$, and $a_* = \frac{1}{4}(b_2 - 3b_1)(b_2 - b_1)^{-2}$ whenever $b_2 \ge 3b_1$. With $f_g(t) = a + (b_1 + b_2)t^2$, it is readily seen that f_g has no fixed point if $a > a^*$, whereas

$$\xi = \frac{2a}{1 + \sqrt{1 - a/a^*}}\,, \quad \eta = 2a^*\big(1 + \sqrt{1 - a/a^*}\big)\,,$$

are fixed points of f_g whenever $a \le a^*$. As a consequence of Theorem 2.2, $A_{+\infty} = \mathbb{I}^2$ if $a > a^*$, and $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}^2$ if $a \le a^*$. These two cases are indicated in Figure 4 by a white and a grey region, respectively. Note that if $a = a^*$ then $\xi = \eta = 2a^*$, and A_ξ is closed. On the other hand, if $a < a^*$ then A_ξ is open (in $\mathbb{I}^2$), and $(\eta, \eta) \in \partial A_\xi$. To understand the dynamics of (4.1) for $(x_1, x_2) \in \partial A_\xi$ in this case, observe that the two

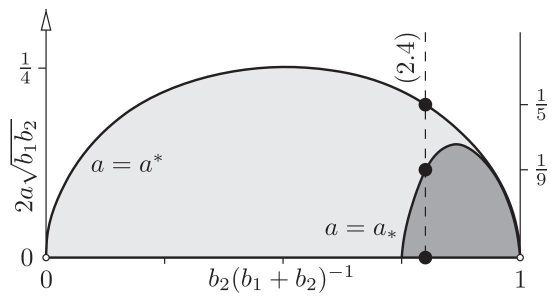

Figure 4. Visualizing the parameter space of (4.1); if $a > a^*$ then $A_{+\infty} = \mathbb{I}^2$ (white region), whereas if $a \le a^*$ then $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}^2$ (grey regions). Parameter values for which (4.1) has a pair of non-constant 2-periodic solutions correspond to the dark-grey region, and a dashed line indicates the special case of (2.4), for which $a_* = \frac{1}{9}$ and $a^* = \frac{1}{5}$.

equations $g(x_1, x_2) = x_2, g(x_2, x_1) = x_1$ have no solution at all for $a > a^*$, have only the obvious solutions $x_1 = x_2 = \xi$ and $x_1 = x_2 = \eta$ for $a_* \le a \le a^*$, but have two additional solutions $(\eta^\pm, \eta^\mp)$ with

$$\eta^\pm = \tfrac{1}{2}(b_2 - b_1)^{-1} \pm 2\sqrt{a^*}\sqrt{a_* - a}\,,$$

whenever $0 \le a < a_*$. Thus $\partial A_\xi = A_\eta$ for $a_* \le a \le a^*$, whereas $\partial A_\xi \ne A_\eta = \{(\eta, \eta)\}$ whenever $a < a_*$. In the latter case, represented by a dark-grey region in Figure 4, for every solution (x_n) of (4.1) with $(x_1, x_2) \in \partial A_\xi$ and $x_1 \ne \eta$,

$$\lim_{n\to\infty} x_{2n-1} = \eta^+ \quad \text{and} \quad \lim_{n\to\infty} x_{2n} = \eta^-$$

or vice versa, depending on whether $x_1 > \eta$ or $x_1 < \eta$. In particular, $(x_n) = (\eta^\pm, \eta^\mp, \eta^\pm, \eta^\mp, \dots)$ are the only two non-constant periodic solutions of (4.1); though unstable, they are *conditionally* attracting, i.e. attracting within ∂A_ξ.

Example 4.2: As a generalization of (1.2), consider the difference equation

$$x_n = b_1 x_{n-1}^{1+\alpha_1} + b_2 x_{n-2}^{1+\alpha_2}\,, \quad \forall n \ge 3\,, \tag{4.2}$$

where $b_1, b_2, \alpha_1, \alpha_2 \in \mathbb{R}^+$. Thus $g(x) = b_1 x_1^{1+\alpha_1} + b_2 x_2^{1+\alpha_2}$ and $g \in \mathcal{G}$, so Theorem 2.2 applies. The associated map $f_g(t) = b_1 t^{1+\alpha_1} + b_2 t^{1+\alpha_2}$ has the fixed points $\xi = 0$ and $\eta > 0$, with η being uniquely determined by $b_1\eta^{\alpha_1} + b_2\eta^{\alpha_2} = 1$. It follows that $\overline{A_0} \cup A_{+\infty} = \mathbb{I}^2$. To study the dynamics of (4.2) on ∂A_0, consider the two equations

$$g(x_1, x_2) = b_1 x_1^{1+\alpha_1} + b_2 x_2^{1+\alpha_2} = x_2\,, \quad g(x_2, x_1) = b_1 x_2^{1+\alpha_1} + b_2 x_1^{1+\alpha_2} = x_1\,. \tag{4.3}$$

Note that (4.3) is invariant under an interchange of x_1 and x_2, and $x_1 x_2 = 0$ entails $x = (0, 0)$. To find all solutions $x \in \mathbb{I}^2$ of (4.3) with $x_1 \ne x_2$, therefore, it suffices to consider the case $x_2 = t x_1 > 0$ with $t > 1$. This yields

$$x_1^{\alpha_1} = \frac{t(t^{\alpha_2} - 1)}{b_1(t^{2+\alpha_1+\alpha_2} - 1)}\,, \quad x_1^{\alpha_2} = \frac{t^{2+\alpha_1} - 1}{b_2(t^{2+\alpha_1+\alpha_2} - 1)}\,,$$

and consequently

$$b_1^{\alpha_2} b_2^{-\alpha_1} = \frac{t^{\alpha_2}(t^{\alpha_2} - 1)^{\alpha_2}}{(t^{2+\alpha_1} - 1)^{\alpha_1}(t^{2+\alpha_1+\alpha_2} - 1)^{\alpha_2-\alpha_1}} =: h_{\alpha_1,\alpha_2}(t) \,. \tag{4.4}$$

The auxiliary function $h_{\alpha_1,\alpha_2} :]1, +\infty[\to \mathbb{R}^+$ is C^∞, with

$$h_{\alpha_1,\alpha_2}(1+) = \frac{\alpha_2^{\alpha_2}}{(2+\alpha_1)^{\alpha_1}(2+\alpha_1+\alpha_2)^{\alpha_2-\alpha_1}} \,, \quad h'_{\alpha_1,\alpha_2}(1+) = 0 \,,$$

and $\lim_{t\to+\infty} h_{\alpha_1,\alpha_2}(t) = 0$. Thus, assuming that h_{α_1,α_2} is decreasing, either $b_1^{\alpha_2} b_2^{-\alpha_1} \geq h_{\alpha_1,\alpha_2}(1+)$, in which case $\partial A_0 = A_\eta$, or else $b_1^{\alpha_2} b_2^{-\alpha_1} < h_{\alpha_1,\alpha_2}(1+)$, in which case $\partial A_0 \neq A_\eta$, and each solution (x_n) of (4.2) with $(x_1, x_2) \in \partial A_0 \setminus \{(\eta, \eta)\}$ is attracted to one of two non-constant 2-periodic solutions. If h_{α_1,α_2} is decreasing, therefore, (4.2) exhibits exactly the two dynamical scenarios observed previously for (4.1) with $a < a^*$.

While it may be tedious to identify *all* $\alpha_1, \alpha_2 > 0$ for which h_{α_1,α_2} is decreasing, some special cases are easy to analyze. For instance, if $\alpha_1 = \alpha_2$ the function

$$h_{\alpha_1,\alpha_1}(t)^{1/\alpha_1} = \frac{t^{1+\alpha_1} - t}{t^{2+\alpha_1} - 1}$$

is readily seen to be decreasing on $]1, +\infty[$, with $h_{\alpha_1,\alpha_1}(1+)^{1/\alpha_1} = \alpha_1(2+\alpha_1)^{-1}$. In case $\alpha_1 = \alpha_2$, therefore, each solution of (4.2) with $(x_1, x_2) \in \partial A_0$ and $x_1 \neq \eta$ either converges to η, or else is attracted to a non-constant 2-periodic solution, depending on whether $b_1 b_2^{-1} \geq \alpha_1(2+\alpha_1)^{-1}$ or not. Note that for $\alpha_1 = \alpha_2 = 1$ the condition $b_1 b_2^{-1} \geq \alpha_1(2+\alpha_1)^{-1} = \frac{1}{3}$ is equivalent to $b_2(b_1 + b_2)^{-1} \leq \frac{3}{4}$, which was encountered already in Example 4.1 for $a = 0$; see also Figure 4.

It is important to notice, however, that the function h_{α_1,α_2} may not be decreasing for all α_1, α_2. To see this, choose for example $\alpha_1 = 10$ and $\alpha_2 = 4$, and hence

$$h_{10,4}(t) = \frac{t^4(t^4+1)^6(t^8+1)^6}{(t^8+t^4+1)^{10}} \,.$$

Note that $h_{10,4}(1) = 2^{12} \cdot 3^{-10}$, $h'_{10,4}(1) = 0$, and $\lim_{t\to+\infty} t^4 h_{10,4}(t) = 1$. The function $h_{10,4}$ is *not* decreasing on $[1, +\infty[$; in fact, with $t^* = \left(2+\sqrt{3}\right)^{1/4} > 1$, it is *increasing* on $[1, t^*]$, decreasing on $[t^*, +\infty[$, and $h_{10,4}(t^*) = 2^{15} \cdot 3^3 \cdot 5^{-10} > h_{10,4}(1)$. It follows that for $h_{10,4}(1) < b_1^4 b_2^{-10} < h_{10,4}(t^*)$, the left equality in (4.4) holds with two different values of $t > 1$. Correspondingly, the difference equation

$$x_n = b_1 x_{n-1}^{11} + b_2 x_{n-2}^5 \,, \quad \forall n \geq 3 \,, \tag{4.5}$$

has *four* different non-constant 2-periodic solutions in this case. In total, (4.5) admits the following possible dynamical scenarios for $(x_1, x_2) \in \partial A_0$, depending on the value of $\beta := b_1^4 b_2^{-10} > 0$; see also Figure 5.

(i) If $\beta > h_{10,4}(t^*)$ then simply $\partial A_0 = A_\eta$.
(ii) If $\beta = h_{10,4}(t^*)$ then (4.5) has two conditionally semi-attracting non-constant 2-periodic solutions.

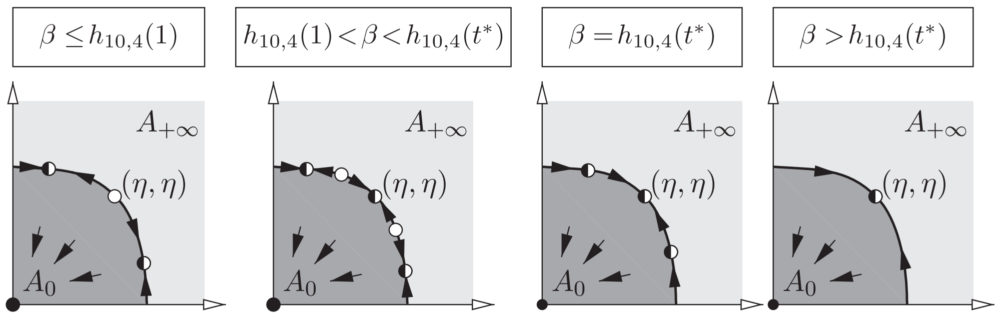

Figure 5. Depending on the value of $\beta = b_1^4 b_2^{-10} > 0$, the solutions of (4.5) exhibit four different dynamical scenarios on ∂A_0; see Example 4.2.

(iii) As seen above, if $h_{10,4}(1) < \beta < h_{10,4}(t^*)$ then (4.5) has four non-constant 2-periodic solutions of which two are conditionally attracting and two are repelling. Also, the constant solution $x_n \equiv \eta$ of (4.5) is conditionally attracting in this case.

(iv) Finally, if $\beta \leq h_{10,4}(1)$ then only two (conditionally attracting) 2-periodic solutions exist, and the constant solution $x_n \equiv \eta$ is repelling.

In the above examples, note that if $b_1 = b_2$ (and $\alpha_1 = \alpha_2$ in Example 4.2) then the recursions (4.1) and (4.2) do not have any non-constant 2-periodic solutions. These are special instances of a more general observation that follows directly from the proof of Theorem 2.2, together with the fact that $f \oplus f \in \mathcal{G}$ for every $f \in \mathcal{F}$.

Proposition 4.3: *Let $f \in \mathcal{F}$. Then for each solution (x_n) of*

$$x_n = f(x_{n-1}) + f(x_{n-2}), \quad \forall n \geq 3,$$

either $x_n \to \xi \in \mathbb{I}$ with $2f(\xi) = \xi$, or else $x_n \to +\infty$.

An inspection of the arguments in Section 3 shows that parts of the conclusion of Theorem 2.2 remain intact if the C^1-function $g : \mathbb{I}^2 \to \mathbb{I}$ has the property that

$$g(\,\cdot\,, x_2),\ g(x_1, \,\cdot\,) \in \mathcal{F}, \quad \forall x \in \mathbb{I}^2. \tag{4.6}$$

Clearly, every $g \in \mathcal{G}$ satisfies (4.6), but the converse is not true. For instance, if $f_1, f_2 \in \mathcal{F}$ and $f_1(0), f_2(0) > 0$ then (4.6) holds for $g = f_1 \odot f_2$, defined as $f_1 \odot f_2(x) = f_1(x_1) f_2(x_2)$ for all $x \in \mathbb{I}^2$, and yet $f_1 \odot f_2$ may not belong to $\mathcal{G}$. (Recall that, by contrast, $f_1 \oplus f_2 \in \mathcal{G}$ whenever $f_1, f_2 \in \mathcal{F}$.)

Example 4.4: Let $f(t) = a + t^2$, with $a \geq 0$, and consider

$$g(x) = f \odot f(x) = (a + x_1^2)(a + x_2^2), \quad \forall x \in \mathbb{I}^2.$$

Clearly, $f \in \mathcal{F}$, yet $g \notin \mathcal{G}$, and hence Theorem 2.2 does not apply. Still, the asymptotic behaviour of (1.1) is much aligned with that theorem. Specifically, it is straightforward to identify a number $0 < a^* < 3 \cdot 2^{-8/3}$ such that for $a \geq a^*$ all conclusions of Theorem 2.2 remain correct: If $a > 3 \cdot 2^{-8/3}$ then simply $A_{+\infty} = \mathbb{I}^2$, whereas if $a^* \leq a \leq 3 \cdot 2^{-8/3}$ then there exists a unique $\xi > 0$, depending on a, with $\overline{A_\xi} \cup A_{+\infty} = \mathbb{I}^2$, where A_ξ is a non-empty bounded convex neighbourhood of $(0, 0)$. In fact, if $a = 3 \cdot 2^{-8/3}$ then

$\xi = 2^{-4/3}$ and $\overline{A_\xi} = A_\xi$, whereas if $a < 3 \cdot 2^{-8/3}$ then $\partial A_\xi = A_\eta$ for a unique $\eta > \xi$. For $0 < a < a^*$ this partition of $\mathbb{I}^2$ persists, however A_ξ no longer is convex. Finally, if $a = 0$ then $g(\,\cdot\,, 0) = g(0, \cdot) = 0 \notin \mathcal{F}$, and thus even (4.6) fails. Nonetheless $\overline{A_0} \cup A_{+\infty} = \mathbb{I}^2$, with $A_0 = \left\{x \in \mathbb{I}^2 : x_1^{\sqrt{3}-1} x_2 < 1\right\}$, and $\partial A_0 = A_1$. Though still a neighbourhood of $(0, 0)$, the set A_0 is neither bounded nor convex.

Difference equations of 'product-type' have been studied e.g. in [4,17,28], though with somewhat different objective. Note that, unlike in the previous example, $g = f_1 \odot f_2$ may belong to $\mathcal{G}$ for *some* $f_1, f_2 \in \mathcal{F}$, in which case Theorem 2.2 does apply directly. For instance, it is easily checked that $f_1 \odot f_2 \in \mathcal{G}$ provided that the (almost everywhere defined) functions $(f_1'/f_1)'$ and $(f_2'/f_2)'$ both are positive. For example, $\cosh(\,\cdot\,^2 + a_1) \odot \cosh(\,\cdot\,^2 + a_2) \in \mathcal{G}$ for all $a_1, a_2 \geq 0$. The following analogue of Proposition 4.3, then, is an immediate consequence of the proof of Theorem 2.2.

Proposition 4.5: *Let $f \in \mathcal{F}$, and assume that $(f'/f)' > 0$. Then for each solution (x_n) of*

$$x_n = f(x_{n-1}) f(x_{n-2}), \quad \forall n \geq 3,$$

either $x_n \to \xi \in \mathbb{I}$ with $f(\xi)^2 = \xi$, or else $x_n \to +\infty$.

Remark 4.6: If $g \in \mathcal{G}$ is real-analytic on $(\mathbb{R}^+)^2$ then (1.1) can have at most finitely many 2-periodic solutions. (All functions g considered as examples in this article are real-analytic.) It seems plausible that the assumption $g \in \mathcal{G}$ does not in general restrict the number of such solutions any further, i.e. in Theorem 2.2(ii) the curve ∂A_ξ may contain any (even) number of non-constant 2-periodic solutions of (1.1). At the time of writing, the authors are not aware of a concrete example where this number is larger than *four*, the latter having been observed in (4.5).

It is natural to ask whether an analogue of Theorem 2.2 holds for higher-order difference equations

$$x_n = g(x_{n-1}, x_{n-2}, \ldots, x_{n-d}), \quad \forall n \geq d + 1, \tag{4.7}$$

with an integer $d \geq 3$. For instance, an ad-hoc analysis along the lines of Section 3 shows that for every $a > 0$, the set $A_0 \subset \mathbb{I}^3$ associated with the third-order equation

$$x_n = x_{n-1}^2 + a x_{n-2}^2 + x_{n-3}^2, \quad \forall n \geq 4, \tag{4.8}$$

is a bounded convex neighbourhood of $(0, 0, 0)$, and $\overline{A_0} \cup A_{+\infty} = \mathbb{I}^3$. For $a \leq 6$, each solution of (4.8) with $(x_1, x_2, x_3) \in \partial A_0$ converges to $(a + 2)^{-1}$, whereas for $a > 6$, most such solutions are attracted to one of two non-constant 2-periodic solutions; the latter are (conditional) sinks for (4.8) within the surface ∂A_0. In general, it is conceivable that the dynamics of (4.7) may allow for more alternatives than Theorem 2.2, even when $d = 3$.

More generally still, one may consider difference equations

$$x_n = g(x_{n-N_1}, x_{n-N_2}, \ldots, x_{n-N_d}), \quad \forall n \geq N_d + 1, \tag{4.9}$$

with fixed positive integers $N_1 < N_2 < \cdots < N_d$ (assumed w.l.o.g. to not have a common factor), of which

$$x_n = x_{n-3}^2 + x_{n-7}^2, \quad \forall n \geq 8,$$

would be a simple example. While the definition of the family $\mathcal{G}$ in (2.3) can be adapted to (4.7) without difficulty, doing the same for (4.9) in a fruitful way seems less straightforward if g is understood as a function on $\mathbb{I}^{N_d}$ rather than on $\mathbb{I}^d$.

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