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Regular and chaotic motion of a kicked pendulum: A Markovian approach

Mechanical devices subject to impulsive excitation may exhibit very complicated dynamics. Though desirable, a complete analysis of the statistical morphogenesis of (the maps induced by) such systems usually is highly demanding. We therefore focus on a special class of maps nevertheless wide enough to comprise a number of interesting examples. Furthermore, an approximation technique tailored to this specific class is shown to improve Markovian approximation techniques discussed in the literature.

1. Statistical stability

Let $(I, \mathcal{B}, \lambda)$ denote the unit interval together with the σ-algebra of its Borel subsets and Lebesgue measure; futhermore, assume that the measurable map $T : I \to I$ be non-singular, i.e. $\lambda\big(T^{-1}(B)\big) = 0$ whenever $\lambda(B) = 0$. The uniquely determined linear operator P_T on L^1 satisfying

$$\int_{T^{-1}(B)} f \, d\lambda = \int_B P_T f \, d\lambda \quad \text{for all } B \in \mathcal{B} \text{ and } f \in L^1$$

is called the *Frobenius-Perron operator* associated with T. This positive, non-expansive operator constitutes a major tool in the statistical analysis of dynamical systems ([1,4]). According to [4] the map T is termed *statistically stable* if there exists a unique P_T-invariant density $f^* \in L^1$ and $P_T^n f \to f^*$ as $n \to \infty$ for every density f. Conditions implying statistical stability (as well as the weaker form of asymptotic periodicity) are extensively studied in the literature. In view of the application below we state the following result which may be considered a slight modification of the classical Lasota-Yorke theorem ([4]).

Theorem *Assume that for a finite number of points* $0 = a_0 < a_1 < \ldots < a_{r-1} < a_r = 1$ *the map* $T : I \to I$ *is* C^2 *on* $]a_{i-1}, a_i[$ *and has a* C^1*-extension to* $[a_{i-1}, a_i]$ *for all* $i = 1, \ldots, r$. *Then* T *is statistically stable if only* $\lim_{x \nearrow a_i} T(x) = 0$ *for all* $i = 1, \ldots, r$ *and* $\sup_{x \in I} T'(x) \le -\tau$ *for some* $\tau > 1$.

In order to deal with the mechanical application below we consider a special class of maps on I. Let $f : I \to [0, \infty[$ be a strictly decreasing C^2 function with $f(1) = 1$ and $\beta > 1$. The map $\tilde{T}$ on $[0, \infty[$ defined as

$$\tilde{T}(x) := \begin{cases} f(x) & \text{if } x \in I, \\ \beta^{-1} x & \text{if } x \notin I, \end{cases}$$

induces a measurable map T on I according to $T(x) := \tilde{T}^{n(x)}(x)$ where $n(x) := \min\{l \in \mathbb{N} \,|\, \tilde{T}^l(x) \in I\}$. A short calculation yields $T(x) = f(x)\beta^{-\lceil \log_\beta f(x) \rceil}$; consequently $T(I) \subseteq [\beta^{-1}, 1]$, and the analysis of T may be restricted to the latter interval.

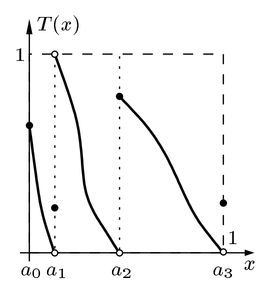

Fig.1 A map satisfying the assumptions of the modified Lasota-Yorke theorem

Corollary *Let* f *and* β *be as above. If the induced map* T *is expanding, i.e.* $\inf_x |T'(x)| > 1$ *on* $[\beta^{-1}, 1]$, *then it is statistically stable.*

2. An example: The kicked pendulum

We shall statistically investigate the dynamics of a kicked *long* pendulum with linear friction (see figure 2). The linearized equation of motion reads $ml^2\ddot{\varphi} + k\dot{\varphi} + mgl\varphi = 0$. In order to keep the pendulum in motion a kick is exerted whenever the pendulum's angular velocity does not exceed ω_0 as the pendulum goes through the vertical position from the right to the left, i.e. whenever $0 < -\dot{\varphi}|_{\varphi=0} < \omega_0$; specifically, we assume that at each kick the angular velocity $\dot{\varphi}^-$ is instantaneously enlarged to $\dot{\varphi}^+ = K(\dot{\varphi}^-) > \dot{\varphi}^-$. After introducing the non-dimensional quantities $x := \dot{\varphi}^-/\omega_0$ as well as $\rho := \frac{k}{2ml^2}\sqrt{\frac{l}{g}}$ and $\sigma := \frac{2\pi\rho}{\sqrt{1-\rho^2}}$ we are more or less in the situation discussed above with $f(x) := K(\omega_0 x)/\omega_0$ and $\beta = e^\sigma$. (For simplicity the damping is assumed to be *weak*, i.e. $0 < \rho < 1$.) Clearly, any measurable map on the unit interval could be obtained in this way by appropriately specifying the kick-law K. In the sequel we shall, however, exclusively deal with the affine rule $f(x) := 1 + \alpha(1 - x)$ where $\alpha > 0$. Intuitively a the parameter α. Depending on the parameters σ and α the dynamics of the resulting map $T_{(\sigma,\alpha)}$ on I may differ quite considerably.

By introducing the boundary functions $b_s^-(\xi) := (e^{s\xi} - e^{\xi})/(e^{\xi} - 1)$ and $b_s^+(\xi) := e^{s\xi}$ $(s \in \mathbb{N})$, it is easy to see that $T_{(\sigma,\alpha)}$ has an attracting fixed point x^* whenever $(\sigma,\alpha) \in B_s := \{(\xi,\eta) \in \mathbb{R}_+^2 | b_s^-(\xi) < \eta < b_s^+(\xi)\}$ for some $s \in \mathbb{N}$. In the latter case $X_s := \{x \in I \,|\, T^n_{(\sigma,\alpha)}(x) \not\to x^*\}$ turns out to be a Cantor set if $s \geq 3$. Its Hausdorff dimension can be calculated as $\dim_H(X_s) = \log_\beta Z$ where Z denotes the unique solution in $[1,\beta]$ of $z + z^2 + \ldots + z^{s-1} = z^{\log_\beta \alpha}$. (If $(\sigma,\alpha) \in B_s$ then $\dim_H(X_s) < 1$; on the boundary b_s^- one finds $\dim_H(X_s) = 1$ while on b_s^+ the relation $\dim_H(X_s) = \frac{\sigma_s^*}{\sigma} \leq 1$ holds, with σ_s^* being uniquely defined by $b_s^-(\sigma_s^*) = b_s^+(\sigma_s^*)$.) Furthermore, $(X_s, T_{(\sigma,\alpha)}|_{X_s})$ is easily seen to be topologically conjugate to the full shift on $s-1$ symbols (cf.[2]). The sets B_s $(s \geq 3)$ thus provide "tongues" of transient chaos (see figure 3).

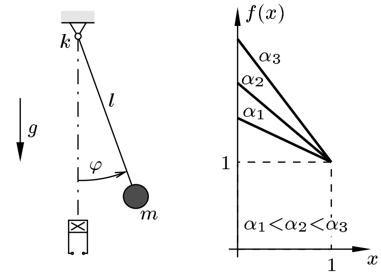

Fig.2 The pendulum model (left) and the non-dimensional kick law f

If $(\sigma,\alpha) \notin \bigcup_{s\in\mathbb{N}} \overline{B}_s$ the map $T_{(\sigma,\alpha)}$ is piecewise expanding and therefore statistically stable by the above corollary. In general the unique $T_{(\sigma,\alpha)}$-invariant density f^* will be rather complicated. (Since $T_{(\sigma,\alpha)}$ is piecewise affine one could write down an explicit formula for f^* which in fact turns out to be hardly illuminating,[3].) Probably the most convenient way of discussing f^* and its morphogenesis under varying (σ,α) consists in approximating $T_{(\sigma,\alpha)}$ by a *Markov map*. By definition, such a map sends each interval $]a_{i-1}, a_i[$ to a union of such intervals. It is well known that the analysis of P_T reduces to a matter of finite-dimensional linear algebra, if T is an expanding, piecewise affine Markov map. In particular, there always exists an invariant density which is piecewise constant. Moreover, a lot of approximation techniques have been discussed in the literature (see [1] and the references cited therein). However, the computational effort due to these methods usually grows *exponentially* with the number of approximation steps that have to be performed. As far as the present problem is concerned, a much better approximation can be found. Although $T_{(\sigma,\alpha)}$ will not be Markovian in general, it might be so with respect to a refined partition: assume that $T^n_{(\sigma,\alpha)}(0) \in \{a_0, \ldots, a_r\}$ for some $n \in \mathbb{N}$. It is easy then to see that $T_{(\sigma,\alpha)}$ is a Markov map with respect to $\{a_0, \ldots, a_r\} \cup \{T_{(\sigma,\alpha)}(0), \ldots, T^{n-1}_{(\sigma,\alpha)}(0)\}$. Consequently, the unique solution of $P_{T_{(\sigma,\alpha)}} f^* = f^*$ can be found by solving a linear equation in $\mathbb{R}^{r+n-2}$. (Observe that the dimension of the latter problem grows *linearly* with n.) For the system under consideration it is easily seen that the set made up by those parameters (σ,α) which give rise to a Markov map is in fact *dense* in $\mathbb{R}_+^2$. It therefore should not come as a surprise that our Markovian analysis provides a rather complete picture of the system's statistical morphogenesis. A few results in this direction are summarized by figure 3. It is worth noting that due to the discontinuities in the family $T_{(\sigma,\alpha)}$ there can be observed more dramatic dynamical changes than for other, more regular families. For example, if α crosses one of the lines b_s^+ from below, an immediate transition from transient to full chaos takes place via a continuum of two-periodic points; following [2] we call this effect a *chaotic explosion*.

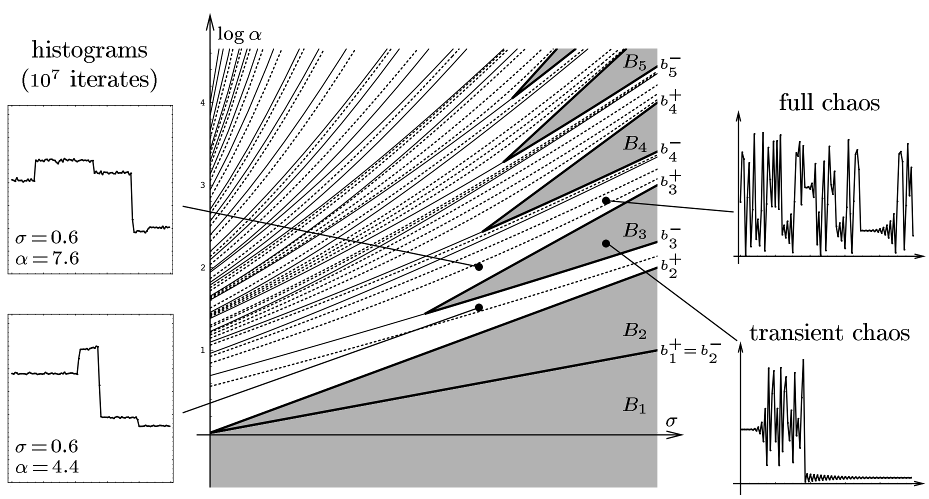

Fig.3 Statistical analysis in the parameter plane; a few first and second order Markovian situations are indicated by solid and broken lines respectively.

3. References

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