A definition of spectrum for differential equations on finite time

A. Berger [a,*], T.S. Doan [b], S. Siegmund [b]

[a] *Mathematical and Statistical Sciences, University of Alberta, Canada*
[b] *Fachrichtung Mathematik, TU Dresden, Germany*

Abstract

Hyperbolicity of an autonomous rest point is characterised by its linearization not having eigenvalues on the imaginary axis. More generally, hyperbolicity of any solution which exists for all times can be defined by means of Lyapunov exponents or exponential dichotomies. We go one step further and introduce a meaningful notion of hyperbolicity for linear systems which are defined for finite time only, i.e. on a compact time interval. Hyperbolicity now describes the transient dynamics on that interval. In this framework, we provide a definition of finite-time spectrum, study its relations with classical concepts, and prove an analogue of the Sacker–Sell spectral theorem: For a d-dimensional system the spectrum is non-empty and consists of at most d disjoint (and often compact) intervals. An example illustrates that the corresponding spectral manifolds may not be unique, which in turn leads to several challenging questions.
© 2008 Elsevier Inc. All rights reserved.

1. Introduction

Spectral theory of autonomous equations $\dot{x} = Bx$ with $B \in \mathbb{R}^{d \times d}$ rests upon the decomposition $E_1 \oplus \cdots \oplus E_n = \mathbb{R}^d$ of $\mathbb{R}^d$ into the sum of n generalized eigenspaces ($1 \leqslant n \leqslant d$). Each E_j,

---

* Corresponding author.
  E-mail addresses: aberger@math.ualberta.ca (A. Berger), dtson@tu-dresden.de (T.S. Doan),
stefan.siegmund@tu-dresden.de (S. Siegmund).

the generalised eigenspace corresponding to all eigenvalues with real part λ_j, is invariant and can, for any small $\varepsilon > 0$ and the appropriate $K = K(\varepsilon) \geqslant 1$, be characterised dynamically as

$$\xi \in E_j \quad \Longleftrightarrow \quad \left\| e^{Bt}\xi \right\| \leqslant Ke^{(\lambda_j \pm \varepsilon)t} \|\xi\| \quad \text{as } t \to \pm\infty.$$

In this article, we are interested in a spectral theory for finite-time, nonautonomous linear equations, i.e., time is confined to a non-empty compact interval $I = [t_-, t_+] \subset \mathbb{R}$, and the differential equation considered may depend explicitly on time,

$$\dot{x} = A(t)x \quad (t \in I), \tag{1}$$

with $A \in L^1_{\text{loc}}(I, \mathbb{R}^{d \times d})$, the space of locally integrable matrix functions $A : I \to \mathbb{R}^{d \times d}$. Let $\Phi : I \times I \to \mathbb{R}^{d \times d}$, $(t, s) \mapsto \Phi(t, s)$ denote the evolution operator associated with (1), that is, $t \mapsto \Phi(t, s)\xi$ solves the initial value problem (1) together with $x(s) = \xi$, for any $s \in I$, $\xi \in \mathbb{R}^d$.

To put our approach into perspective, we first recall related concepts for unbounded I, specifically for $I = \mathbb{R}$. Note that the (time-dependent) eigenvalues of A are generally[1] not related to growth rates of solutions. This can for instance be seen from the periodic matrix

$$A(t) = \begin{bmatrix} -1 - 2\cos 4t & 2 + 2\sin 4t \\ -2 + 2\sin 4t & -1 + 2\cos 4t \end{bmatrix},$$

which has a constant double eigenvalue -1, yet $t \mapsto e^t \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$ is an unbounded solution of (1). In the case of periodic equations, Floquet multipliers provide an adequate substitute for the (real parts of) eigenvalues. It is well known, however, that Floquet theory cannot be extended beyond the periodic case (see [7]).

For $I = \mathbb{R}$ and general A, the Lyapunov exponent $\lambda(\xi) := \limsup_{t \to +\infty} \frac{1}{t} \log \|\Phi(t, 0)\xi\|$ for $\xi \in \mathbb{R}^d \setminus \{0\}$ measures the exponential growth rate of a solution of (1) as $t \to +\infty$, since for every $\varepsilon > 0$ there exists $K = K(\varepsilon) \geqslant 1$ with

$$\left\| \Phi(t, 0)\xi \right\| \leqslant Ke^{(\lambda(\xi) + \varepsilon)t} \|\xi\| \quad \forall t \geqslant 0.$$

Note that if $A(t) \equiv B$ then the Lyapunov exponents are exactly the real parts of the eigenvalues of B. Since any two linearly dependent solutions of (1) have the same Lyapunov exponent, in general there exist at most d different Lyapunov exponents for (1). If for instance all Lyapunov exponents are negative then, with some $\alpha > 0$ and $K = K(s) \geqslant 1$,

$$\left\| \Phi(t, s) \right\| \leqslant Ke^{-\alpha(t-s)} \quad \forall t \geqslant s, \tag{2}$$

which shows that (1) is asymptotically stable. However, [6, pp. 321–322] gives an example of a nonlinearly perturbed system $\dot{x} = A(t)x + f(t, x)$ where $\|f(t, x)\| \leqslant C\|x\|^{1+\alpha}$ with $C, \alpha > 0$, for which the zero solution is unstable. Thus negativity of all Lyapunov exponents is not sufficient to deduce asymptotic stability for small nonlinear perturbations. If, on the other hand, (2) holds with K not depending on s, then asymptotic stability for small nonlinear perturbation follows, a result often referred to as linearized asymptotic stability for nonautonomous systems [1]. With K

---

[1] If $\frac{d}{dt}A(t)$ is small enough then the (slowly) varying eigenvalues of A may still describe growth rates of solutions, see e.g. [5].

independent of s, (2) is in fact a special case of an exponential dichotomy (see below), and consequently the extended state space $I \times \mathbb{R}^d$ can be split into linear integral manifolds consisting of (uniformly) exponentially decaying and growing solutions, respectively. Recall that $M \subset I \times \mathbb{R}^d$ is a *linear integral manifold* of (1) if M is *invariant*, i.e., $(s, \xi) \in M$ implies $(t, \Phi(t,s)\xi) \in M$ for all $t \in I$, and for every $t \in I$ the *fiber* $M(t) = \{\xi \in \mathbb{R}^d \colon (t, \xi) \in M\}$ is a linear subspace of $\mathbb{R}^d$. Since $\Phi(t,s)$ is an isomorphism, the fibers have constant dimension. Trivially, $I \times \mathbb{R}^d$ and $I \times \{0\}$ are linear integral manifolds. Every linear integral manifold is a topological manifold in $I \times \mathbb{R}^d$ and a vector bundle over I. If M_1 and M_2 are linear integral manifolds of (1), then so are their *intersection* and *sum*,

$$M_1 \cap M_2 := \left\{(t, \xi) \in I \times \mathbb{R}^d \colon \xi \in M_1(t) \cap M_2(t)\right\},$$

$$M_1 + M_2 := \left\{(t, \xi) \in I \times \mathbb{R}^d \colon \xi \in M_1(t) + M_2(t)\right\}.$$

We write $M_1 \oplus M_2$ if, for every $t \in I$, the sum $M_1(t) + M_2(t)$ is direct. Every splitting of $I \times \mathbb{R}^d$ into a direct sum of linear integral manifolds corresponds to an *invariant projector* of (1), that is, a projection-valued function $P : I \to \mathbb{R}^{d \times d}$ satisfying

$$P(t)\Phi(t,s) = \Phi(t,s)P(s) \quad \forall t, s \in I. \tag{3}$$

Note that P is continuous due to the identity $P(t) \equiv \Phi(t,s)P(s)\Phi(s,t)$. The rank of $P(t)$ is independent of $t \in I$, and we define $\mathrm{rk}\, P := \mathrm{rk}\, P(t_-)$. Image $\mathrm{im}\, P := \{(t, \xi) \in I \times \mathbb{R}^d \colon \xi \in \mathrm{im}\, P(t)\}$ and kernel $\ker P := \{(t, \xi) \in I \times \mathbb{R}^d \colon \xi \in \ker P(t)\}$ of P are clearly linear integral manifolds of (1), leading to the splitting $\ker P \oplus \mathrm{im}\, P = I \times \mathbb{R}^d$.

With these preparations, to introduce hyperbolicity recall that (1) admits an *exponential dichotomy* (ED) if there exists an invariant projector P, together with constants $K \geqslant 1$ and $\alpha > 0$, such that

$$\left\|\Phi(t,s)P(s)\right\| \leqslant K e^{-\alpha(t-s)} \quad \forall t \geqslant s,$$

$$\left\|\Phi(t,s)\left[\mathrm{id}_{d \times d} - P(s)\right]\right\| \leqslant K e^{\alpha(t-s)} \quad \forall t \leqslant s. \tag{4}$$

For example, an autonomous system $\dot{x} = Bx$ admits an ED if and only if B is hyperbolic, i.e. has no eigenvalues on the imaginary axis. In the case $I = \mathbb{R}$ the existence of an ED is the appropriate notion of hyperbolicity for (1), and the corresponding spectrum is the *dichotomy* (or *Sacker–Sell*) *spectrum*

$$\Sigma_{\mathrm{dich}}(A) = \left\{\gamma \in \mathbb{R} \colon \dot{x} = \left(A(t) - \gamma\, \mathrm{id}_{d \times d}\right)x \text{ does not admit an ED}\right\}.$$

The structure of $\Sigma_{\mathrm{dich}}(A)$ is clarified by

Theorem 1. *(See [8, Spectral theorem for $I = \mathbb{R}$].) Assume that $\|\Phi(t,s)\| \leqslant K e^{a|t-s|}$ for all $t, s \in \mathbb{R}$, with some $K, a > 0$. Then the dichotomy spectrum $\Sigma_{\mathrm{dich}}(A)$ of (1) is the non-empty disjoint union of at most d compact (possibly one-point) intervals, called spectral intervals, i.e.*

$$\Sigma_{\mathrm{dich}}(A) = [a_1, b_1] \cup \cdots \cup [a_n, b_n],$$

where $1 \leqslant n \leqslant d$. Associated with the spectral intervals are uniquely determined linear integral manifolds $\mathcal{W}_1, \dots, \mathcal{W}_n$ (called spectral manifolds) satisfying

$$\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n = \mathbb{R} \times \mathbb{R}^d. \tag{5}$$

In the autonomous case $\dot{x} = Bx$ we have $\Sigma_{\mathrm{dich}}(B) = \{\Re\lambda : \lambda \text{ is an eigenvalue of } B\} = \{\lambda_1, \dots, \lambda_n\}$ and $\mathcal{W}_j = \mathbb{R} \times E_j$, where E_j is the generalized eigenspace corresponding to all eigenvalues with real part λ_j.

The purpose of this article is to provide meaningful definitions of hyperbolicity and spectrum for (1) on the finite-time interval $I = [t_-, t_+]$. It is mainly motivated by the recent surge in research activity on *finite-time dynamics* and its wide range of potential applications (see e.g. [2] and the many references therein). A new notion of hyperbolicity is discussed in Section 2. It is modelled upon exponential dichotomies but, since asymptotic concepts are meaningless for bounded I, it describes the transient behaviour instead. In Section 3 we define finite-time growth rates and use them to provide two characterizations of hyperbolicity. Our main result is Theorem 17, a spectral theorem in the spirit of Theorem 1 for (1) on $I = [t_-, t_+]$. Unlike for $I = \mathbb{R}$, spectral manifolds need not be unique for bounded I. In a final section we illustrate this fact by means of a simple example and discuss some of its implications.

2. Finite-time hyperbolicity

In the case $I = \mathbb{R}$, existence of an ED for (1) is an adequate, well established notion of hyperbolicity. If, however, I is a compact interval then the estimates (4) can trivially be satisfied for *every* invariant projector P, simply by choosing K sufficiently large. Thus the concept of an ED captures (uniform) *asymptotic* exponential growth rates only and is inappropriate for describing finite-time behaviour. To get an idea as to how the latter could be dealt with, recall first that by the equivalence of norms it can be assumed that the norm in (4) is an induced matrix norm, and hence (4) can be rewritten as

$$\left\| \Phi(t, t_-)\xi \right\| \leqslant K e^{-\alpha(t-s)} \left\| \Phi(s, t_-)\xi \right\| \quad \forall \xi \in \operatorname{im} P(t_-),\ t \geqslant s,$$

$$\left\| \Phi(t, t_-)\xi \right\| \leqslant K e^{\alpha(t-s)} \left\| \Phi(s, t_-)\xi \right\| \quad \forall \xi \in \ker P(t_-),\ t \leqslant s. \tag{6}$$

Unlike in the case $I = \mathbb{R}$, the actual value of K does play a role for finite-time considerations. The subsequent discussion will make it clear that no reasonable notion of hyperbolicity can possibly be based upon (6) if K differs from its minimal possible value. Thus it will be assumed that $K = 1$. With this, it follows from (6) that the function $t \mapsto \|\Phi(t, t_-)\xi\|$ is strictly (in fact, exponentially) decreasing for every $\xi \in \operatorname{im} P(t_-)$ and is strictly increasing for every $\xi \in \ker P(t_-)$. This is exactly what we want to capture as the constituting feature of finite-time hyperbolicity: the existence of solutions exhibiting, uniformly on I, monotone exponential growth or decay w.r.t. some norm. Clearly, this approach depends crucially on the chosen norm. Therefore, throughout this article, we let $\Gamma \in \mathbb{R}^{d \times d}$ denote a symmetric positive definite matrix, that is, $\Gamma = \Gamma^\top > 0$, and symbolise by $\| \cdot \|_\Gamma$ the induced norm, i.e., $\|x\|_\Gamma = \sqrt{\langle x, \Gamma x \rangle}$ for all $x \in \mathbb{R}^d$, with $\langle \cdot, \cdot \rangle$ representing the standard inner product. Quantities depending on Γ have their dependence made explicit by a subscript which is suppressed only if Γ equals $\operatorname{id}_{d \times d}$, the $d \times d$ identity matrix. Also, from now on I denotes the interval $[t_-, t_+]$ with $-\infty < t_- < t_+ < +\infty$.

Definition 2 *(Hyperbolicity).* Eq. (1) is *hyperbolic* (*on I and w.r.t. the Γ-norm*) if there exists an invariant projector P, together with constants $\alpha < 0 < \beta$, such that for all $t, s \in I$

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{\alpha(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \mathrm{im}\, P(s),\ t \geqslant s,$$

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{\beta(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \ker P(s),\ t \leqslant s. \tag{7}$$

Remark 3. (i) If (1) is hyperbolic with constants $\alpha < 0 < \beta$ then (7) is also satisfied with constants $\hat{\alpha}, \hat{\beta}$ whenever $\alpha \leqslant \hat{\alpha} < 0 < \hat{\beta} \leqslant \beta$.

(ii) The hyperbolicity estimates (7) are equivalent to

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{\alpha(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \mathrm{im}\, P(s),\ t \geqslant s,$$

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \geqslant e^{\beta(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \ker P(s),\ t \geqslant s. \tag{8}$$

While the invariant projector associated with an ED on $\mathbb{R}$ is uniquely determined [5], this may not be the case for the projector P according to Definition 2.

Example 4. (i) Consider (1) with $A(t) \equiv \mathrm{diag}[1, -1]$ and evolution operator $\Phi(t,s) = \mathrm{diag}[e^{t-s}, e^{-t+s}]$. It is readily confirmed that the one-parameter family of projectors, parametrised by $\theta \in \mathbb{R}$,

$$P_\theta(t) = \begin{bmatrix} \sin^2\theta & e^{2t}\sin\theta\cos\theta \\ e^{-2t}\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

is invariant. Moreover, the estimates (7) hold with $\Gamma = \mathrm{id}_{2\times2}$, $P = P_\theta$, and $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$, provided that $|\tan\theta| \leqslant \frac{1}{2}\min\{e^{2t_-}, e^{-2t_+}\}$. In the latter case, therefore, (1) is hyperbolic on I with any invariant projector P_θ. Note that $\theta \in \pi\mathbb{Z}$ would follow, and hence P_θ would be uniquely determined (and constant), if either $t_- = -\infty$ or $t_+ = +\infty$, that is, if I was unbounded.

(ii) Similarly to (i) consider (1) with $A(t) \equiv \mathrm{diag}[1, -1, -2]$, so that $\Phi(t,s) = \mathrm{diag}[e^{t-s}, e^{-t+s}, e^{-2(t-s)}]$. Again it is easy to check that

$$P_\delta(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \delta e^{-3t} & 0 & 1 \end{bmatrix}$$

is invariant for any $\delta \in \mathbb{R}$. The estimates (7) now hold with $\Gamma = \mathrm{id}_{3\times3}$, $P = P_\delta$, and $\alpha = -1$, $\beta = 1$, and consequently (1) is hyperbolic on I with associated projector P_δ, provided that $|\delta| \leqslant \frac{1}{2}e^{3t_-}$. In this example the projector would only be unique if I was unbounded *to the left*, i.e. for $t_- = -\infty$. This asymmetry is due to the fact that $\mathrm{im}\, P_\delta$ does not depend on δ.

Although the invariant projector of a hyperbolic system may not be unique, as seen in Example 4, its rank is nevertheless unique.

Lemma 5. *Assume that* (1) *is hyperbolic with associated projector P according to Definition 2. Let $\widehat{P}$ be any invariant projector such that, for all $t, s \in I$,*

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{\hat\alpha(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \operatorname{im} \widehat{P}(s),\ t \geqslant s,$$

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{\hat\beta(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \ker \widehat{P}(s),\ t \leqslant s, \tag{9}$$

with constants $\hat\alpha < 0 < \hat\beta$. Then $\operatorname{rk} \widehat{P} = \operatorname{rk} P$.

Proof. Assume by way of contradiction that $\operatorname{rk} P(t_-) > \operatorname{rk} \widehat{P}(t_-)$. Since $\operatorname{rk} P(t_-) + \operatorname{rk}(\operatorname{id}_{d\times d} - \widehat{P}(t_-)) > d$, there exists a non-zero $\xi \in \operatorname{im} P(t_-) \cap \ker \widehat{P}(t_-)$. From (8) we deduce that

$$e^{\hat\beta(t-t_-)} \|\xi\|_\Gamma \leqslant \left\| \Phi(t,t_-)\xi \right\|_\Gamma \leqslant e^{\alpha(t-t_-)} \|\xi\|_\Gamma$$

for all $t \geqslant t_-$. Since $\alpha < 0 < \hat\beta$ this implies that $\|\xi\|_\Gamma = 0$, contradicting $\xi \neq 0$. Hence $\operatorname{rk} P(t_-) \leqslant \operatorname{rk} \widehat{P}(t_-)$. Similarly $\operatorname{rk} P(t_-) \geqslant \operatorname{rk} \widehat{P}(t_-)$, and thus $\operatorname{rk} P(t_-) = \operatorname{rk} \widehat{P}(t_-)$. $\quad\square$

In the following, the family of equations

$$\dot{x} = \big(A(t) - \gamma \operatorname{id}_{d\times d}\big)x \quad (t \in I), \tag{10}$$

parametrised by $\gamma \in \mathbb{R}$, will play an important role (see also [8]). It is straightforward to check that $\Phi_\gamma(t,s) := e^{-\gamma(t-s)}\Phi(t,s)$ is the corresponding evolution operator. If $\dot{x} = (A(t) - \gamma \operatorname{id}_{d\times d})x$ is hyperbolic then the associated projector P is also invariant for $\dot{x} = A(t)x$. Consequently, the hyperbolicity estimates for (10) are equivalent to

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{(\gamma+\alpha)(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \operatorname{im} P(s),\ t \geqslant s,$$

$$\left\| \Phi(t,s)\xi \right\|_\Gamma \leqslant e^{(\gamma+\beta)(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \ker P(s),\ t \leqslant s. \tag{11}$$

Remark 6. If $\dot{x} = (A(t) - \gamma \operatorname{id}_{d\times d})x$ is hyperbolic with associated projector $P(t) \equiv \operatorname{id}_{d\times d}$ (or $P(t) \equiv 0$) then $\dot{x} = (A(t) - \zeta \operatorname{id}_{d\times d})x$ is also hyperbolic with the same projector for every $\zeta \geqslant \gamma$ (or $\zeta \leqslant \gamma$).

3. Growth rates and spectrum

We describe the finite-time behaviour of (1) by means of extremal growth rates taken over appropriate subspaces. Evidently, every linear subspace $X \subset \mathbb{R}^d$ induces the linear integral manifold $\{(t, \Phi(t,t_-)\xi) \in I \times \mathbb{R}^d \colon t \in I,\ \xi \in X\}$ consisting of all solutions starting in X at time t_-. Also, recall that, for any $\xi \in X$, the function $t \mapsto \|\Phi_\gamma(t,t_-)\xi\|_\Gamma$ is absolutely continuous and hence its derivative is defined for almost all $t \in I$.

Definition 7 (*Growth rates*)*.* Let $X \subset \mathbb{R}^d$, $X \neq \{0\}$ denote a linear subspace and let $\gamma \in \mathbb{R}$. Then, with the evolution operator $\Phi_\gamma(t,t_-)$ associated with (10),

$$\underline{\lambda}_\gamma(X) := \inf\left\{ \operatorname*{ess\,inf}_{t\in I} \frac{\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_\gamma(t,t_-)\xi\|_\Gamma}{\|\Phi_\gamma(t,t_-)\xi\|_\Gamma} \colon \xi \in X,\ \xi \neq 0 \right\}$$

and

$$\bar{\lambda}_\gamma(X) := \sup\left\{ \operatorname*{ess\,sup}_{t \in I} \frac{\frac{\mathrm{d}}{\mathrm{d}t} \|\Phi_\gamma(t, t_-)\xi\|_\Gamma}{\|\Phi_\gamma(t, t_-)\xi\|_\Gamma} : \xi \in X,\ \xi \neq 0 \right\}$$

are, respectively, the *lower* and *upper* (*essential*) *growth rate* of X (or the integral manifold induced by X). For notational convenience we define $\underline{\lambda}_\gamma(\{0\}) := +\infty$ and $\bar{\lambda}_\gamma(\{0\}) := -\infty$.

Remark 8. (i) As with Φ, we suppress the subscript γ whenever $\gamma = 0$, i.e., we write $\underline{\lambda}(X)$ instead of $\underline{\lambda}_0(X)$, etc.

(ii) $\underline{\lambda}_\zeta(X) = \underline{\lambda}_\gamma(X) + \gamma - \zeta$ and $\bar{\lambda}_\zeta(X) = \bar{\lambda}_\gamma(X) + \gamma - \zeta$ for all $\gamma, \zeta \in \mathbb{R}$, so that in particular $\underline{\lambda}_\gamma(X) = \underline{\lambda}(X) - \gamma$ and $\bar{\lambda}_\gamma(X) = \bar{\lambda}(X) - \gamma$.

(iii) Obviously, $\underline{\lambda}_\gamma(X) \leqslant \bar{\lambda}_\gamma(X)$ for any X, and $X \subset Y$ implies that $\underline{\lambda}_\gamma(X) \geqslant \underline{\lambda}_\gamma(Y)$ and $\bar{\lambda}_\gamma(X) \leqslant \bar{\lambda}_\gamma(Y)$.

(iv) If $\underline{\lambda}_\gamma(X) > 0$ then $\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma > 0$ for almost all $t \in I$ and $\xi \in X \setminus \{0\}$, i.e., $t \mapsto \|\Phi_\gamma(t, t_-)\xi\|_\Gamma$ is increasing. Similarly, if $\bar{\lambda}_\gamma(X) < 0$ then $t \mapsto \|\Phi_\gamma(t, t_-)\xi\|_\Gamma$ is decreasing.

Lemma 9 *(First characterization of hyperbolicity). Eq. (10) is hyperbolic, with invariant projector P and constants $\alpha < 0 < \beta$, if and only if*

$$\bar{\lambda}_\gamma\big(\operatorname{im} P(t_-)\big) \leqslant \alpha \quad \text{and} \quad \underline{\lambda}_\gamma\big(\ker P(t_-)\big) \geqslant \beta. \tag{12}$$

Proof. Assume first that (10) is hyperbolic with projector P and constants $\alpha < 0 < \beta$. Remark 3(ii), with ξ replaced by $\Phi_\gamma(s, t_-)\xi$, shows that the function $t \mapsto e^{-\alpha t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma$ is non-increasing for every $\xi \in \operatorname{im} P(t_-)$ whereas $t \mapsto e^{-\beta t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma$ is non-decreasing for every $\xi \in \ker P(t_-)$. Consequently, for almost all $t \in I$,

$$0 \geqslant e^{\alpha t}\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{-\alpha t} \|\Phi_\gamma(t, t_-)\xi\|_\Gamma \right) = \frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma - \alpha\|\Phi_\gamma(t, t_-)\xi\|_\Gamma \quad \forall \xi \in \operatorname{im} P(t_-),$$

$$0 \leqslant e^{\beta t}\left(\frac{\mathrm{d}}{\mathrm{d}t} e^{-\beta t} \|\Phi_\gamma(t, t_-)\xi\|_\Gamma \right) = \frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma - \beta\|\Phi_\gamma(t, t_-)\xi\|_\Gamma \quad \forall \xi \in \ker P(t_-),$$

and hence $\bar{\lambda}_\gamma(\operatorname{im} P(t_-)) \leqslant \alpha$ as well as $\underline{\lambda}_\gamma(\ker P(t_-)) \geqslant \beta$, i.e., (12) holds.

Conversely, if (12) holds then, for almost all $t \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma - \alpha\|\Phi_\gamma(t, t_-)\xi\|_\Gamma \leqslant 0 \quad \forall \xi \in \operatorname{im} P(t_-),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi_\gamma(t, t_-)\xi\|_\Gamma - \beta\|\Phi_\gamma(t, t_-)\xi\|_\Gamma \geqslant 0 \quad \forall \xi \in \ker P(t_-),$$

from which (11) follows by integration. Hence (10) is hyperbolic. $\quad\square$

Definition 10 *(Extremal k-dimensional growth rates).* For every $k \in \{0, 1, \ldots, d\}$ and $\gamma \in \mathbb{R}$ the numbers $\underline{\lambda}_\gamma^{(k)}$ and $\bar{\lambda}_\gamma^{(k)}$ are, respectively, the *maximal k-dimensional lower growth rate* and the *minimal k-dimensional upper growth rate* of (10), that is,

$$\underline{\lambda}_\gamma^{(k)} = \sup_{\dim X = k} \underline{\lambda}_\gamma(X) \quad \text{and} \quad \bar{\lambda}_\gamma^{(k)} = \inf_{\dim X = k} \bar{\lambda}_\gamma(X).$$

(Note that $\underline{\lambda}_\gamma^{(0)} = +\infty$ and $\bar{\lambda}_\gamma^{(0)} = -\infty$, according to Definition 7.)

Remark 11. (i) $\underline{\lambda}_\zeta^{(k)} = \underline{\lambda}_\gamma^{(k)} + \gamma - \zeta = \underline{\lambda}^{(k)} - \zeta$ and $\bar{\lambda}_\zeta^{(k)} = \bar{\lambda}_\gamma^{(k)} + \gamma - \zeta = \bar{\lambda}^{(k)} - \zeta$ for all $k \in \{1, \ldots, d\}$ and $\gamma, \zeta \in \mathbb{R}$.

(ii) For every $\gamma \in \mathbb{R}$, the extremal growth rates are ordered as

$$\underline{\lambda}_\gamma^{(1)} \geqslant \underline{\lambda}_\gamma^{(2)} \geqslant \cdots \geqslant \underline{\lambda}_\gamma^{(d)} \quad \text{and} \quad \bar{\lambda}_\gamma^{(1)} \leqslant \bar{\lambda}_\gamma^{(2)} \leqslant \cdots \leqslant \bar{\lambda}_\gamma^{(d)}.$$

Lemma 12 (*Second characterization of hyperbolicity*). *Eq.* (10) *is hyperbolic with an invariant projector of rank* $k \in \{0, 1, \ldots, d\}$ *if and only if*

$$\bar{\lambda}_\gamma^{(k)} < 0 < \underline{\lambda}_\gamma^{(d-k)}, \tag{13}$$

or, equivalently, if $\bar{\lambda}^{(k)} < \gamma < \underline{\lambda}^{(d-k)}$.

Proof. First assume that (10) is hyperbolic with constants $\alpha < 0 < \beta$ and an invariant projector P of rank k, i.e., $\dim \operatorname{im} P(t_-) = k$ and $\dim \ker P(t_-) = d - k$. Then, in view of Lemma 9 and Definition 10,

$$\bar{\lambda}_\gamma^{(k)} \leqslant \bar{\lambda}_\gamma\big(\operatorname{im} P(t_-)\big) \leqslant \alpha \quad \text{and} \quad \underline{\lambda}_\gamma^{(d-k)} \geqslant \underline{\lambda}_\gamma\big(\ker P(t_-)\big) \geqslant \beta,$$

which proves (13).

Assume in turn that (13) holds for some $k \in \{0, 1, \ldots, d\}$. Hence there exist subspaces X and Y, with $\dim X = k$ and $\dim Y = d - k$, such that $\bar{\lambda}_\gamma(X) < 0 < \underline{\lambda}_\gamma(Y)$. By Remark 8(iv), every solution starting in $X \setminus \{0\}$ is decreasing and every solution starting in $Y \setminus \{0\}$ is increasing (w.r.t. the Γ-norm). This implies that $X \cap Y = \{0\}$, or equivalently, $X \oplus Y = \mathbb{R}^d$. Let $P(t_-)$ be the projection onto X along Y, that is, with $\operatorname{im} P(t_-) = X$, $\ker P(t_-) = Y$, and define an invariant projector $P : I \to \mathbb{R}^{d \times d}$ of rank k by setting $P(t) = \Phi(t, t_-) P(t_-) \Phi(t_-, t)$. Lemma 9 implies that (10) is hyperbolic with projector P and constants $\alpha = \bar{\lambda}_\gamma(X) < 0$ and $\beta = \underline{\lambda}_\gamma(Y) > 0$. $\quad\square$

We are now in a position to introduce a finite-time notion of spectrum for (1).

Definition 13. The *spectrum* of (1) on I and w.r.t. the Γ-norm is the set

$$\Sigma(A) := \Sigma_\Gamma^I(A) := \big\{ \gamma \in \mathbb{R} : \dot{x} = \big(A(t) - \gamma \operatorname{id}_{d \times d} \big) x \text{ is not hyperbolic} \big\};$$

its complement $\rho(A) = \mathbb{R} \setminus \Sigma(A)$ is called the *resolvent set* of (1).

It is obvious from (11) that $\rho(A)$ is open, and hence $\Sigma_\Gamma^I(A)$ is closed. By its very definition, the spectrum $\Sigma_\Gamma^I(A)$ may depend on I as well as on Γ, and it usually does so in a non-trivial way, even if (1) is autonomous.

Example 14. Let $A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ and $\Gamma = \operatorname{id}_{2 \times 2}$. It is straightforward to check that

$$\dot{x} = (A - \gamma \operatorname{id}_{2 \times 2}) x = \begin{bmatrix} 1 - \gamma & -2 \\ 0 & -1 - \gamma \end{bmatrix} x \tag{14}$$

is hyperbolic on any bounded interval $I = [t_-, t_+]$ and w.r.t. $\Gamma = \mathrm{id}_{2 \times 2}$, provided that $\gamma^2 \leqslant 1$ or $\gamma^2 > 2$. If $\gamma^2 = 2$ then (14) is not hyperbolic. A lengthy yet elementary calculation confirms that for $1 < \gamma^2 < 2$ the system (14) is hyperbolic if and only if

$$2(t_+ - t_-) < \left| \log \frac{\gamma - \sqrt{2 - \gamma^2}}{\gamma + \sqrt{2 - \gamma^2}} \right|.$$

Consequently, the spectrum is the union of two non-degenerate intervals,

$$\Sigma_{\mathrm{id}_{2 \times 2}}^I(A) = \left\{ \gamma \in \mathbb{R} : 2 / \left(1 + \tanh^2(t_+ - t_-) \right) \leqslant \gamma^2 \leqslant 2 \right\}.$$

On the other hand, choose $\Gamma = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ and let X_η denote the one-dimensional space spanned by the unit vector η. From

$$\underline{\lambda}(X_\eta) = \frac{(\eta_1 - \eta_2)^2 - \eta_2^2}{(\eta_1 - \eta_2)^2 + \eta_2^2}, \qquad \bar{\lambda}(X_\eta) = \frac{(\eta_1 - \eta_2)^2 - \tau^2 \eta_2^2}{(\eta_1 - \eta_2)^2 + \tau^2 \eta_2^2},$$

where $\tau = e^{-2(t_+ - t_-)}$, it follows that $\underline{\lambda}^{(1)} = \max_\eta \underline{\lambda}(X_\eta) = 1$, $\bar{\lambda}^{(1)} = \min_\eta \bar{\lambda}(X_\eta) = -1$, and $\underline{\lambda}^{(2)} = -1$, $\bar{\lambda}^{(2)} = 1$, so that, for all $\gamma \in \mathbb{R}$,

$$\underline{\lambda}_\gamma^{(1)} = 1 - \gamma, \quad \bar{\lambda}_\gamma^{(1)} = -1 - \gamma \quad \text{as well as} \quad \underline{\lambda}_\gamma^{(2)} = -1 - \gamma, \quad \bar{\lambda}_\gamma^{(2)} = 1 - \gamma. \tag{15}$$

Lemma 12, together with (15), shows that $\Sigma_\Gamma^I(A) = \{-1, 1\}$ for every interval I.

Remark 15. While the dichotomy spectrum is invariant under any constant linear change of coordinates, i.e. $\Sigma_{\mathrm{dich}}(A) = \Sigma_{\mathrm{dich}}(C^{-1} A C)$ for every invertible $C \in \mathbb{R}^{d \times d}$, it follows from the easily checked identity $\Sigma_\Gamma^I(A) = \Sigma_{C^\top \Gamma C}^I(C^{-1} A C)$ that the finite-time spectrum Σ_Γ^I is C-invariant only if C preserves the Γ-norm.

The following basic properties of Σ_Γ^I are analogues of well-known facts about Σ_{dich} (see e.g. Lemma 3.2 in [8]).

Lemma 16. *For every $\gamma \in \rho(A)$ let P_γ be an invariant projector satisfying* (11). *Then*:

 (i) *the map $\gamma \mapsto \mathrm{rk}\, P_\gamma$ is non-decreasing*;
(ii) *if $\gamma_1 < \gamma_2$ then $\mathrm{rk}\, P_{\gamma_1} = \mathrm{rk}\, P_{\gamma_2}$ if and only if $[\gamma_1, \gamma_2] \cap \Sigma_\Gamma^I(A) = \emptyset$.*

Proof. Note that, according to Lemma 5, $\mathrm{rk}\, P_\gamma$ is unambiguously defined for every $\gamma \in \rho(A)$ even though P_γ may not be unique.

To see (i), let $\gamma_1 < \gamma_2$ and pick $\xi \in \mathrm{im}\, P_{\gamma_1} \cap \ker P_{\gamma_2}$. Then, for all $t \geqslant s$,

$$e^{(\gamma_2 + \beta_2)(t-s)} \|\xi\|_\Gamma \leqslant \left\| \Phi(t, s) \xi \right\|_\Gamma \leqslant e^{(\gamma_1 + \alpha_1)(t-s)} \|\xi\|_\Gamma,$$

where $\alpha_1 < 0 < \beta_2$, and hence $\|\xi\|_\Gamma \leqslant e^{(\gamma_1 - \gamma_2 + \alpha_1 - \beta_2)(t-s)} \|\xi\|_\Gamma$. Since $\gamma_1 + \alpha_1 < \gamma_2 + \beta_2$ this implies that $\xi = 0$ and therefore $\mathrm{rk}\, P_{\gamma_1} \leqslant \mathrm{rk}\, P_{\gamma_2}$.

To prove that between any two $\gamma_1 < \gamma_2$ with rk $P_{\gamma_1} <$ rk P_{γ_2} there lies some spectral point, let $\gamma^* := \inf\{\gamma: \text{rk } P_\gamma > \text{rk } P_{\gamma_1}\}$ and assume w.l.o.g. that $\gamma_1 < \gamma^* < \gamma_2$. For all sufficiently large $n \in \mathbb{N}$ either one of the numbers $\gamma^* \pm n^{-1}$ belongs to $[\gamma_1, \gamma_2] \cap \Sigma_\Gamma^I(A)$ or else rk $P_{\gamma^*-n^{-1}} <$ rk $P_{\gamma^*+n^{-1}}$. In the former case, the proof is complete whereas in the latter case there exists ξ_n with $\|\xi_n\|_\Gamma = 1$, together with $\alpha_n < 0 < \beta_n$, such that for all $t \geqslant s$

$$\left\|\Phi_{\gamma^*+n^{-1}}(t,s)\xi_n\right\|_\Gamma \leqslant e^{\alpha_n(t-s)}\|\xi_n\|_\Gamma \quad \text{and} \quad \left\|\Phi_{\gamma^*-n^{-1}}(t,s)\xi_n\right\|_\Gamma \geqslant e^{\beta_n(t-s)}\|\xi_n\|_\Gamma.$$

Passing to a subsequence if necessary, it can be assumed that $\lim_{n\to\infty} \xi_n = \xi^*$ for some ξ^* with $\|\xi^*\|_\Gamma = 1$. Thus, for all $t, s \in I$ with $t \geqslant s$,

$$1 \leqslant \limsup_{n\to\infty} e^{\beta_n(t-s)}\|\xi_n\|_\Gamma \leqslant \left\|\Phi_{\gamma^*}(t,s)\xi^*\right\|_\Gamma \leqslant \liminf_{n\to\infty} e^{\alpha_n(t-s)}\|\xi_n\|_\Gamma \leqslant 1,$$

which in turn shows that (10) is not hyperbolic for $\gamma = \gamma^*$, and hence $\gamma^* \in \Sigma_\Gamma^I(A)$.

To complete the proof of (ii), assume that $\gamma_1 < \gamma_2$ yet rk $P_{\gamma_1} =$ rk P_{γ_2} and pick any $\gamma \in [\gamma_1, \gamma_2]$. Since $\text{im} P_{\gamma_1}(t) \cap \ker P_{\gamma_2}(t) = \{0\}$ for every $t \in I$, the projection onto $\text{im} P_{\gamma_1}(t)$ along $\ker P_{\gamma_2}(t)$ is well defined; denote this projection by $P(t)$. It is readily confirmed that P is an invariant projector for Φ_γ. For all $t \geqslant s$, and with some $\alpha_1 < 0 < \beta_2$,

$$\left\|\Phi_\gamma(t,s)\xi\right\|_\Gamma = e^{(\gamma_1-\gamma)(t-s)}\left\|\Phi_{\gamma_1}(t,s)\xi\right\|_\Gamma \leqslant e^{(\gamma_1-\gamma+\alpha_1)(t-s)}\|\xi\|_\Gamma$$

whenever $\xi \in \text{im } P(s) = \text{im } P_{\gamma_1}(s)$, but also

$$\left\|\Phi_\gamma(t,s)\xi\right\|_\Gamma = e^{(\gamma_2-\gamma)(t-s)}\left\|\Phi_{\gamma_2}(t,s)\xi\right\|_\Gamma \geqslant e^{(\gamma_2-\gamma+\beta_2)(t-s)}\|\xi\|_\Gamma$$

whenever $\xi \in \ker P(s) = \ker P_{\gamma_2}(s)$. Therefore $\gamma \in \rho(A)$, and since γ has been arbitrary, $[\gamma_1, \gamma_2] \cap \Sigma_\Gamma^I(A) = \emptyset$. $\square$

If $A(t)$ is defined for all t then for every sufficiently long interval I the spectrum $\Sigma_\Gamma^I(A)$ captures essentially all the non-hyperbolicity encoded in $\Sigma_{\text{dich}}(A)$. This observation will be made precise in Theorem 19 below. Note, however, that while the structural complexity of $\Sigma_{\text{dich}}(A)$ is severely limited by Theorem 1, it is conceivable from Definition 13 that $\Sigma_\Gamma^I(A)$ could be a much more complicated set. Fortunately though, this is not the case.

Theorem 17 (*Spectral theorem*). *The spectrum $\Sigma(A)$ of (1) is the non-empty disjoint union of at most d intervals* (*called spectral intervals*),

$$\Sigma(A) = [a_1, b_1] \cup \cdots \cup [a_n, b_n], \tag{16}$$

where $1 \leqslant n \leqslant d$ and $-\infty \leqslant a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_n \leqslant b_n \leqslant +\infty$. Moreover, $a_j \in \{\underline{\lambda}^{(1)}, \ldots, \underline{\lambda}^{(d)}\}$ and $b_j \in \{\bar{\lambda}^{(1)}, \ldots, \bar{\lambda}^{(d)}\}$ for every $j = 1, \ldots, n$, that is, the left and right boundary points of $\Sigma(A)$ are given, respectively, by maximal lower and minimal upper growth rates.

Proof. According to Lemma 16, the function

$$r: \begin{cases} \rho(A) \to \{0, 1, \ldots, d\}, \\ \gamma \mapsto \text{rk } P_\gamma \end{cases}$$

is non-decreasing, and $r^{-1}(\{k\})$ is an open (possibly empty or infinite) interval for every $k \in \{0, 1, \ldots, d\}$. It follows from Lemma 12 that

$$\gamma \in r^{-1}(\{k\}) \quad \Longleftrightarrow \quad \bar{\lambda}_\gamma^{(k)} < 0 < \underline{\lambda}_\gamma^{(d-k)} \quad \Longleftrightarrow \quad \bar{\lambda}^{(k)} < \gamma < \underline{\lambda}^{(d-k)},$$

which shows that $r^{-1}(\{k\})$ equals $]\bar{\lambda}^{(k)}, \underline{\lambda}^{(d-k)}[$; note that these intervals are disjoint. Thus the resolvent set $\rho(A)$ of (1) is the disjoint union of d open intervals (some of which may be empty),

$$\rho(A) = \bigcup_{k=0}^{d}]\bar{\lambda}^{(k)}, \underline{\lambda}^{(d-k)}[.$$

Recall that $\bar{\lambda}^{(0)} = -\infty$, $\underline{\lambda}^{(0)} = +\infty$, and consequently $\rho(A)$ consists of up to $d - 1$ open intervals of finite length, together with at most two semi-infinite intervals. The spectrum $\Sigma(A)$ therefore consists of $1 \leqslant n \leqslant d$ closed (possibly one-point) intervals of which at most two are unbounded, i.e., $\Sigma(A)$ has the form (16). Moreover, for every $j \in \{1, \ldots, n\}$ the spectral interval $[a_j, b_j]$ equals $[\underline{\lambda}^{(d-k)}, \bar{\lambda}^{(k+1)}]$ for some $1 \leqslant k < d$.

It remains to show that $\Sigma(A)$ is not empty. To this end, assume by way of contradiction that $\Sigma(A) = \emptyset$. Then rk P_γ would have the same value for all $\gamma \in \mathbb{R}$. If rk $P_\gamma > 0$ then, for every $m \in \mathbb{N}$, there existed $\xi_m \in \mathbb{R}^d$ with $\|\xi_m\|_\Gamma = 1$ such that $\|\Phi(t_+, t_-)\xi_m\|_\Gamma \leqslant e^{-m(t_+-t_-)}$. Assume w.l.o.g. that $\lim_{m\to\infty} \xi_m = \xi^*$ exists and deduce that $\|\Phi(t_+, t_-)\xi^*\|_\Gamma = 0$, which clearly contradicts $\|\xi^*\|_\Gamma = 1$. A similar contradiction arises if rk $P_\gamma = 0$, showing that it is impossible to have $\rho(A) = \mathbb{R}$. Hence $\Sigma(A)$ is not empty. $\quad\square$

Example 21 below shows that every form of $\Sigma(A)$ compatible with (16) does actually occur. Also, as is the case with $\Sigma_{\text{dich}}(A)$, a bounded growth condition is required to ensure that $\Sigma(A)$ is compact. (Note that this condition is trivially met if A is continuous.)

Corollary 18. *If $A \in L^\infty(I, \mathbb{R}^{d\times d})$ then $\Sigma(A)$ is the union of at most d compact (possibly one-point) intervals.*

Proof. Pick $c_\Gamma > 0$ such that $\|By\|_\Gamma \leqslant c_\Gamma \|B\| \|y\|_\Gamma$ holds for all $y \in \mathbb{R}^d$ and $B \in \mathbb{R}^{d\times d}$, and let $f(t) := \|\Phi_\gamma(t, s)x\|_\Gamma$ with $x \neq 0$. From the obvious estimate, valid for all $t \in I$,

$$\left| \dot{f}(t) + \gamma f(t) \right| = \left| \frac{\frac{d}{dt} f^2(t)}{2f(t)} + \gamma f(t) \right| = \left| \frac{\langle \Phi_\gamma(t, s)x, \Gamma(A(t) - \gamma \, \mathrm{id}_{d\times d})\Phi_\gamma(t, s)x \rangle}{f(t)} + \gamma f(t) \right|$$

$$= \left| \frac{\langle \Phi_\gamma(t, s)x, \Gamma A(t)\Phi_\gamma(t, s)x \rangle}{f(t)} \right| \leqslant \frac{f(t)c_\Gamma \|A(t)\| f(t)}{f(t)} = c_\Gamma \|A(t)\| f(t),$$

together with $\|A\|_\infty := \operatorname{ess\,sup}_{t\in I} \|A(t)\| < +\infty$, it follows that, for all $t \geqslant s$,

$$f(s)e^{-(c_\Gamma\|A\|_\infty+\gamma)(t-s)} \leqslant f(t) \leqslant f(s)e^{(c_\Gamma\|A\|_\infty-\gamma)(t-s)}.$$

Consequently, $\Sigma(A) \subset [-c_\Gamma\|A\|_\infty, c_\Gamma\|A\|_\infty]$ and hence $\Sigma(A)$ is bounded. $\quad\square$

If $A(t)$ is defined for all $t \in \mathbb{R}$ then $\Sigma_\Gamma^I(A)$ makes sense for every compact interval I. The rest of this section is devoted to clarifying the relation of these spectra with $\Sigma_{\text{dich}}(A)$. To this end,

fix Γ and denote by $\mathcal{F}$ the family of all compact (non-degenerate) intervals $I \subset \mathbb{R}$. Since $\mathcal{F}$ is directed by inclusion, and since $\Sigma_\Gamma^I(A) \subset \Sigma_\Gamma^J(A)$ whenever $I \subset J$, it is natural to introduce the set

$$\Sigma_\Gamma^{\mathbb{R}}(A) := \overline{\lim_{\mathcal{F}} \Sigma_\Gamma^I(A)} = \overline{\bigcup_{I \in \mathcal{F}} \Sigma_\Gamma^I(A)}. \qquad (17)$$

Example 14 shows that $\Sigma_\Gamma^I(A)$ and $\Sigma_{\text{dich}}(A)$ may be disjoint for each $I \in \mathcal{F}$, even if A is constant. The family $(\Sigma_\Gamma^I(A))_{I \in \mathcal{F}}$ does, however, capture $\Sigma_{\text{dich}}(A)$ through its limit.

Theorem 19. *If $A(t)$ is defined for all $t \in \mathbb{R}$, then $\Sigma_\Gamma^{\mathbb{R}}(A) \supset \Sigma_{\text{dich}}(A)$.*

Proof. Since the statement is trivial otherwise, assume that $\Sigma_\Gamma^{\mathbb{R}}(A) \neq \mathbb{R}$ and pick γ and $\varepsilon > 0$ such that $[\gamma - \varepsilon, \gamma + \varepsilon] \cap \Sigma_\Gamma^{\mathbb{R}}(A) = \emptyset$. The proof amounts to showing that $\gamma \notin \Sigma_{\text{dich}}(A)$.

Since $[\gamma - \varepsilon, \gamma + \varepsilon] \cap \Sigma_\Gamma^I(A)$ is empty, there exists, for every $I \in \mathcal{F}$, an invariant projector $P_I : I \to \mathbb{R}^{d \times d}$, together with constants $\alpha_I < 0 < \beta_I$, such that

$$\left\| \Phi(t, s)\xi \right\|_\Gamma \leqslant e^{(\gamma - \varepsilon + \alpha_I)(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \operatorname{im} P_I(s),$$

whenever $t, s \in I$ and $t \geqslant s$, but also

$$\left\| \Phi(t, s)\xi \right\|_\Gamma \leqslant e^{(\gamma + \varepsilon + \beta_I)(t-s)} \|\xi\|_\Gamma \quad \forall \xi \in \ker P_I(s),$$

whenever $t \leqslant s$. Indeed, as in the proof of Lemma 16(ii), P_I can be chosen as the projection onto the image of some $P_{\gamma - \varepsilon}$ along the kernel of some $P_{\gamma + \varepsilon}$. Lemma 5 implies that $\operatorname{rk} P_I$ has the same value for every interval I, say $\operatorname{rk} P_I = k \in \{0, 1, \ldots, d\}$ for all $I \in \mathcal{F}$. For every $n \in \mathbb{N}$, let $I_n := [-n, n]$ and consider the two linear subspaces $X_n := \operatorname{im} P_{I_n}(0)$ and $Y_n := \ker P_{I_n}(0)$ whose dimensions (k and $d - k$, respectively) are independent of n and which therefore can be assumed to converge (w.r.t. the Grassmann topologies in $G_{k,d}$ and $G_{d-k,d}$, see e.g. [4]). Let $X := \lim_{n \to \infty} X_n$, $Y := \lim_{n \to \infty} Y_n$. It is straightforward to check that $X \cap Y = \{0\}$, and also that, for every $t \in \mathbb{R}$,

$$\lim_{n \to \infty} \operatorname{im} P_{I_n}(t) = \Phi_\gamma(t, 0)X, \qquad \lim_{n \to \infty} \ker P_{I_n}(t) = \Phi_\gamma(t, 0)Y.$$

For every $t \in \mathbb{R}$, let $Q(t)$ denote the projection onto $\Phi_\gamma(t, 0)X$ along $\Phi_\gamma(t, 0)Y$. Again it is easy to verify that Q is an invariant projector. Since, for any $\xi \in \mathbb{R}^d$, $Q(s)\xi \in \Phi_\gamma(s, 0)X = \lim_{n \to \infty} \operatorname{im} P_{I_n}(s)$, there exists a sequence (ξ_n) with $\xi_n \in \operatorname{im} P_{I_n}(s)$ for all n and $\lim_{n \to \infty} \xi_n = Q(s)\xi$. Consequently, for all $t \geqslant s$,

$$\left\| \Phi_\gamma(t, s)Q(s)\xi \right\|_\Gamma = \lim_{n \to \infty} \left\| \Phi_\gamma(t, s)\xi_n \right\|_\Gamma \leqslant \liminf_{n \to \infty} e^{(-\varepsilon + \alpha_{I_n})(t-s)} \|\xi_n\|_\Gamma$$

$$\leqslant e^{-\varepsilon(t-s)} \|\xi_n\|_\Gamma.$$

A completely analogous argument shows that, for all $t \leqslant s$,

$$\left\| \Phi_\gamma(t, s)\big(\operatorname{id}_{d \times d} - Q(s)\big)\xi \right\|_\Gamma \leqslant e^{\varepsilon(t-s)} \|\xi\|_\Gamma.$$

Overall, (10) admits an ED, and therefore $\gamma \notin \Sigma_{\text{dich}}(A)$. $\quad \square$

Remark 20. As evidenced by Example 14, Theorem 19 does not hold in general without the closure being taken in (17).

It is natural to ask whether equality can hold in Theorem 19, at least approximately, for the appropriate Γ. This would amount to choosing Γ in such a way that, perhaps up to a small set, $\Sigma_{\Gamma}^{I}(A) \subset \Sigma_{\mathrm{dich}}(A)$ for all $I \in \mathcal{F}$.

Example 21. In the one-dimensional case, i.e. for $d = 1$ and $A(t) = [a(t)]$, the evolution operator can be given explicitly as $\Phi(t, s) = e^{\int_{s}^{t} a(r)\,\mathrm{d}r}$. From this it is straightforward to deduce that $\Sigma_{\mathrm{dich}}(A) = [\sigma^{-}(a), \sigma^{+}(a)]$, where

$$\sigma^{+}(a) := \inf\left\{\delta \in \mathbb{R}:\ \sup_{t \geqslant s}\left(\int_{s}^{t} a(r)\,\mathrm{d}r - \delta(t - s)\right) < +\infty\right\},$$

$$\sigma^{-}(a) := \sup\left\{\delta \in \mathbb{R}:\ \inf_{t \geqslant s}\left(\int_{s}^{t} a(r)\,\mathrm{d}r - \delta(t - s)\right) > -\infty\right\},$$

with the usual conventions $\inf \emptyset := +\infty$, $\sup \emptyset := -\infty$. Since the choice of Γ is irrelevant for $d = 1$, for every compact interval $I \subset \mathbb{R}$,

$$\Sigma_{\Gamma}^{I}(A) = \left[\operatorname*{ess\,inf}_{t \in I} a(t),\ \operatorname*{ess\,sup}_{t \in I} a(t)\right].$$

Suppose for instance that a is monotone with (possibly infinite) limits $a_{-} := \lim_{t \to -\infty} a(t)$ and $a_{+} := \lim_{t \to +\infty} a(t)$. In this case

$$\Sigma_{\mathrm{dich}}(A) = \left[\min\{a_{-}, a_{+}\}, \max\{a_{-}, a_{+}\}\right] = \Sigma_{\Gamma}^{\mathbb{R}}(A),$$

i.e., equality holds in Theorem 19. To ensure this equality, monotonicity can sometimes be dispensed with, as the example $a : t \mapsto \sin(\log(1 + t^2))$ shows for which $\Sigma_{\mathrm{dich}}(A) = [-1, 1] = \Sigma_{\Gamma}^{\mathbb{R}}(A)$, yet a is oscillatory. If, however, a is almost periodic then $\Sigma_{\mathrm{dich}}(A) = \{\lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} a(s)\,\mathrm{d}s\}$ is a singleton, and hence $\Sigma_{\mathrm{dich}}(A)$ and $\Sigma_{\Gamma}^{\mathbb{R}}(A)$ differ substantially unless a is constant. Overall, $\Sigma_{\mathrm{dich}}(A) = \Sigma_{\Gamma}^{\mathbb{R}}(A)$ holds in the case $d = 1$ if and only if

$$\sigma^{+}(a) = \operatorname*{ess\,sup}_{t \in \mathbb{R}} a(t) \quad \text{and} \quad \sigma^{-}(a) = \operatorname*{ess\,inf}_{t \in \mathbb{R}} a(t).$$

Establishing (let alone, characterising) equality of the two spectra in higher dimensions remains a formidable challenge.

In general, therefore, no choice of Γ may lead to equality in Theorem 19. If, however, A is constant then such a choice is possible generically, and up to any prescribed error in the Hausdorff distance d_H. Recall that $\Sigma_{\mathrm{dich}}(A) = \{\Re\lambda:\ \lambda \text{ is an eigenvalue of } A\}$ whenever A is constant.

Theorem 22. *Let A in* (1) *be constant. Then the following holds*:

(i) *if A is diagonalisable (over $\mathbb{C}$) then there exists $\Gamma = \Gamma^\top > 0$ such that $\Sigma_\Gamma^I(A) = \Sigma_{\mathrm{dich}}(A)$ for all $I \in \mathcal{F}$;*

(ii) *for every $\varepsilon > 0$ there exists Γ such that $d_H(\Sigma_\Gamma^I(A), \Sigma_{\mathrm{dich}}(A)) < \varepsilon$ for all $I \in \mathcal{F}$.*

Proof. To prove (i), let $A = C \operatorname{diag}\big[\lambda_l, \operatorname{diag}\big[\begin{smallmatrix} \mu_m & -\nu_m \\ \nu_m & \mu_m \end{smallmatrix}\big]\big]C^{-1}$ with the appropriate regular matrix $C \in \mathbb{R}^{d \times d}$, where λ_l are the real, and $\mu_m \pm i\nu_m$ are the non-real eigenvalues of A. Then

$$\Phi(t, s) = e^{(t-s)A} = C \operatorname{diag}\left[e^{(t-s)\lambda_l}, e^{(t-s)\mu_m} \operatorname{diag}\begin{bmatrix} \cos\nu_m(t-s) & -\sin\nu_m(t-s) \\ \sin\nu_m(t-s) & \cos\nu_m(t-s) \end{bmatrix}\right]C^{-1},$$

and choosing Γ such that $C^\top \Gamma C = \operatorname{id}_{d \times d}$, that is, $\Gamma = (C^{-1})^\top C^{-1}$, leads to

$$\big\|\Phi(t, s)\xi\big\|_\Gamma^2 = \big\langle C^{-1}\xi, \operatorname{diag}\big[e^{2(t-s)\lambda_l}, e^{2(t-s)\mu_m}\operatorname{id}_{2\times 2}\big]C^{-1}\xi\big\rangle \tag{18}$$

for all $t, s \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$. For every $\gamma \in \mathbb{R}$, let X_γ and Y_γ denote the sum of the generalised (realified) eigenspaces corresponding to eigenvalues λ of A with $\Re\lambda < \gamma$ and $\Re\lambda > \gamma$, respectively, that is,

$$X_\gamma := \bigoplus_{\Re\lambda < \gamma} \big(\ker(A - \lambda\operatorname{id}_{d\times d})^d\big)_\mathbb{R}, \qquad Y_\gamma := \bigoplus_{\Re\lambda > \gamma} \big(\ker(A - \lambda\operatorname{id}_{d\times d})^d\big)_\mathbb{R}.$$

If $\gamma \in \mathbb{R} \setminus \Sigma_{\mathrm{dich}}(A)$ then $X_\gamma \oplus Y_\gamma = \mathbb{R}^d$, and with $P(s) \equiv P$ chosen as the (constant) projection onto X_γ along Y_γ, it follows from (18) that, for all $t \geqslant s$,

$$\big\|\Phi_\gamma(t, s)\xi\big\|_\Gamma \leqslant e^{\alpha_\gamma(t-s)}\sqrt{\langle C^{-1}\xi, C^{-1}\xi\rangle} = e^{\alpha_\gamma(t-s)}\|\xi\|_\Gamma,$$

whenever $\xi \in \operatorname{im} P(s) = X_\gamma$; here $\alpha_\gamma := -\gamma + \max\{\delta \in \Sigma_{\mathrm{dich}}(A): \delta < \gamma\} < 0$. Similarly, $\|\Phi_\gamma(t, s)\xi\|_\Gamma \geqslant e^{\beta_\gamma(t-s)}\|\xi\|_\Gamma$ holds with $\beta_\gamma := -\gamma + \min\{\delta \in \Sigma_{\mathrm{dich}}(A): \delta > \gamma\} > 0$, for all $t \geqslant s$ and $\xi \in \ker P(s) = Y_\gamma$. Thus (10) is hyperbolic on I and w.r.t. Γ whenever $\gamma \notin \Sigma_{\mathrm{dich}}(A)$, i.e. $\Sigma_\Gamma^I(A) \subset \Sigma_{\mathrm{dich}}(A)$. On the other hand, for any $\gamma \in \Sigma_{\mathrm{dich}}(A)$ let

$$Z_\gamma := \bigoplus_{\Re\lambda = \gamma} \big(\ker(A - \lambda\operatorname{id}_{d\times d})^d\big)_\mathbb{R}$$

and observe that $\|\Phi_\gamma(t, s)\xi\|_\Gamma = \|\xi\|_\Gamma$ for all $t, s \in I$ and $\xi \in Z_\gamma$. Consequently, if (10) was hyperbolic with some invariant projector P then $\operatorname{rk} P \leqslant d - \dim(Z_\gamma + Y_\gamma)$, but also $d - \operatorname{rk} P = \operatorname{rk}(\operatorname{id}_{d\times d} - P) \leqslant d - \dim(Z_\gamma + X_\gamma)$; this implies that $\dim(Z_\gamma + X_\gamma) + \dim(Z_\gamma + Y_\gamma) \leqslant d$, which obviously is impossible. Thus $\gamma \in \Sigma_\Gamma^I(A)$, and overall $\Sigma_\Gamma^I(A) = \Sigma_{\mathrm{dich}}(A)$ for each $I \in \mathcal{F}$. This completes the proof of (i).

As the subsequent argument can be applied individually to each block in the *real* Jordan normal form of A, it is enough to verify (ii) for a single Jordan block, corresponding to an eigenvalue $\mu \pm i\nu$. For notational convenience replace ε by 2ε in the statement of (ii), and assume

that d is even and replace it by $2d$; the symbol $J_d(a)$ will be used throughout to denote, for any $a \in \mathbb{R}$, the Jordan block

$$J_d(a) = \begin{bmatrix} a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & a & 1 \\ 0 & \cdots & \cdots & 0 & a \end{bmatrix} = a \operatorname{id}_{d \times d} + J_d(0).$$

Given $\varepsilon > 0$, choose an invertible matrix $C_\varepsilon \in \mathbb{R}^{2d \times 2d}$ such that

$$A = C_\varepsilon \begin{bmatrix} \mu \operatorname{id}_{d \times d} + \varepsilon J_d(0) & -\nu \operatorname{id}_{d \times d} \\ \nu \operatorname{id}_{d \times d} & \mu \operatorname{id}_{d \times d} + \varepsilon J_d(0) \end{bmatrix} C_\varepsilon^{-1},$$

and consequently, for all $t, s \in \mathbb{R}$,

$$\Phi_\gamma(t, s) = e^{(\mu - \gamma)(t - s)} C_\varepsilon Q \left(\sum_{k=0}^{d-1} \frac{\varepsilon^k (t - s)^k}{k!} \operatorname{diag}\left[J_d(0)^k, J_d(0)^k \right] \right) C_\varepsilon^{-1},$$

where $Q \in \mathbb{R}^{2d \times 2d}$ denotes the orthogonal matrix

$$Q = \begin{bmatrix} \operatorname{id}_{d \times d} \cos \nu(t - s) & -\operatorname{id}_{d \times d} \sin \nu(t - s) \\ \operatorname{id}_{d \times d} \sin \nu(t - s) & \operatorname{id}_{d \times d} \cos \nu(t - s) \end{bmatrix}.$$

Choose Γ such that $C_\varepsilon^\top \Gamma C_\varepsilon = \operatorname{id}_{2d \times 2d}$. Then, for all $t, s \in \mathbb{R}$ and $\xi \in \mathbb{R}^{2d}$,

$$\left\| \Phi_\gamma(t, s) \xi \right\|_\Gamma = e^{(\mu - \gamma)(t - s)} \left\| \sum_{k=0}^{d-1} \frac{\varepsilon^k (t - s)^k}{k!} \operatorname{diag}\left[J_d(0)^k, J_d(0)^k \right] C_\varepsilon^{-1} \xi \right\|$$

$$\leqslant e^{(\mu - \gamma)(t - s)} \left\| C_\varepsilon^{-1} \xi \right\| \sum_{k=0}^{d-1} \frac{\varepsilon^k |t - s|^k}{k!} \leqslant e^{(\mu - \gamma)(t - s) + \varepsilon |t - s|} \| \xi \|_\Gamma.$$

Thus, if $\gamma \geqslant \mu + 2\varepsilon$ then $\| \Phi_\gamma(t, s) \xi \|_\Gamma \leqslant e^{-\varepsilon(t - s)} \| \xi \|_\Gamma$ for all $t \geqslant s$ and all $\xi \in \mathbb{R}^{2d}$. Similarly, if $\gamma \leqslant \mu - 2\varepsilon$ then $\| \Phi_\gamma(t, s) \xi \|_\Gamma \leqslant e^{\varepsilon(t - s)} \| \xi \|_\Gamma$ for all $t \leqslant s$. For every γ with $|\gamma - \mu| \geqslant 2\varepsilon$ therefore (10) is hyperbolic, and so $\Sigma_\Gamma^I(A) \subset {]}\mu - 2\varepsilon, \mu + 2\varepsilon{[}$. Moreover, since $\operatorname{rk} P_{\gamma + 2\varepsilon} = 2d > 0 = \operatorname{rk} P_{\gamma - 2\varepsilon}$, Lemma 16(ii) implies that $\Sigma_\Gamma^I(A) \cap {]}\mu - 2\varepsilon, \mu + 2\varepsilon{[} \neq \emptyset$, and therefore μ is less than 2ε away from some point in $\Sigma_\Gamma^I(A)$.

Repeating the above argument for each block in the real Jordan normal form of A yields a matrix Γ (depending on ε) such that, for every $I \in \mathcal{F}$,

$$\Sigma_\Gamma^I(A) \subset \bigcup_{\mu \in \Sigma_{\text{dich}}(A)} {]}\mu - 2\varepsilon, \mu + 2\varepsilon{[}$$

but also

$$\Sigma_{\text{dich}}(A) \subset \left\{ r \in \mathbb{R}: \exists \gamma \in \Sigma_\Gamma^I(A) \text{ s.t. } |\gamma - r| < 2\varepsilon \right\}.$$

Overall, therefore, $d_H(\Sigma_\Gamma^I(A), \Sigma_{\text{dich}}(A)) < 2\varepsilon$ for all $I \in \mathcal{F}$. $\quad\square$

Corollary 23. *If A is constant then*

$$\inf_{\Gamma = \Gamma^\top > 0} d_H\left(\Sigma_\Gamma^{\mathbb{R}}(A), \Sigma_{\text{dich}}(A)\right) = 0, \tag{19}$$

and the infimum is attained whenever A is diagonalisable (over $\mathbb{C}$).

Example 24. To see that Theorem 22 is in a way best possible, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = J_2(0)$ and thus $\Sigma_{\text{dich}}(A) = \{0\}$. It will now be shown that $\gamma = 0$ does, somewhat surprisingly, *not* belong to $\Sigma_\Gamma^I(A)$ for any $I \in \mathcal{F}$ and any $\Gamma = \Gamma^\top > 0$. To this end, define the smooth positive function $f : t \mapsto \|te_1 + e_2\|_\Gamma^2$, where e_1, e_2 symbolise the first and second vector in the canonical basis of $\mathbb{R}^2$, respectively. From

$$\left| \dot{f}(t) - 2t\|e_1\|_\Gamma^2 \right| = 2\left| \langle e_1, \Gamma e_2 \rangle \right| \leqslant 2\|e_1\|_\Gamma \|e_2\|_\Gamma,$$

it follows that f is strictly increasing (decreasing) whenever $t \geqslant T$ ($t \leqslant -T$) with $T := 2\|e_2\|_\Gamma / \|e_1\|_\Gamma$. Given $I = [t_-, t_+]$, denote for every $s \in I$ by $P(s)$ the projection onto the one-dimensional space spanned by $(s - t_+ - T)e_1 + e_2$, along the space spanned by $(s - t_- + T)e_1 + e_2$. Clearly, P is an invariant projector and, for every $\xi \in \operatorname{im} P(s)$ and all $t, s \in I$ with $t \geqslant s$,

$$\left\| \Phi(t, s)\xi \right\|_\Gamma = \|\xi\|_\Gamma \sqrt{\frac{f(t - t_+ - T)}{f(s - t_+ - T)}} \leqslant \|\xi\|_\Gamma \sqrt{1 - \frac{2\|e_1\|_\Gamma \|e_2\|_\Gamma}{f(s - t_+ - T)}(t - s)}$$

$$\leqslant \|\xi\|_\Gamma \sqrt{1 - \frac{2\|e_1\|_\Gamma \|e_2\|_\Gamma}{f(t_- - t_+ - T)}(t - s)} \leqslant e^{\alpha_I(t - s)} \|\xi\|_\Gamma,$$

where $\alpha_I = -\|e_1\|_\Gamma \|e_2\|_\Gamma / f(t_- - t_+ - T) < 0$. A similar computation shows that $\|\Phi(t, s)\xi\|_\Gamma \geqslant e^{\beta_I(t-s)}\|\xi\|_\Gamma$ holds for every $\xi \in \ker P(s)$ and all $t \geqslant s$, where $\beta_I = \frac{1}{2}(t_+ - t_-)^{-1} \log(1 + 2\|e_1\|_\Gamma \|e_2\|_\Gamma (t_+ - t_-)/f(t_+ - t_- + T)) > 0$. Therefore, $\dot{x} = Ax$ is hyperbolic on any interval $I \in \mathcal{F}$ and w.r.t. any Γ-norm, i.e. $0 \notin \Sigma_\Gamma^I(A)$.

To elucidate further the specific form of the spectrum for the present example, let $\gamma^+ := \max_{\|x\|_\Gamma = 1} \langle x, \Gamma Ax \rangle$ as well as $\gamma^- := \min_{\|x\|_\Gamma = 1} \langle x, \Gamma Ax \rangle$. A straightforward computation shows that $\gamma^\pm = \pm\gamma_0$, where $\gamma_0 = \frac{1}{2}\langle e_1, \Gamma e_1 \rangle / \sqrt{\det \Gamma} > 0$. Define two one-dimensional subspaces X^+ and X^- according to

$$X^\pm := \left\{ x \in \mathbb{R}^2: \langle x, \Gamma Ax \rangle = \pm\gamma_0 \|x\|_\Gamma^2 \right\}.$$

Since $t \mapsto \|\Phi_{\gamma^+}(t, s)\xi\|_\Gamma$ is non-increasing for every $\xi \in \mathbb{R}^2$, if γ^+ belonged to $\rho(A)$ and had P as an associated invariant projector, then necessarily $\operatorname{rk} P = 2$. But this is impossible because $\frac{d}{dt}\|\Phi_{\gamma^+}(t, s)\xi\|_\Gamma = 0$ for all $\xi \in X^+$. Thus $\gamma^+ \in \Sigma_\Gamma^I(A)$ and, by an analogous argument, also $\gamma^- \in \Sigma_\Gamma^I(A)$. Note further that $\langle \Phi_{\gamma^\pm}(t, s)\xi, \frac{d}{dt}\Phi_{\gamma^\pm}(t, s)\xi \rangle = 0$ whenever $\Phi_{\gamma^\pm}(t, s)\xi \in X^\pm$, while $\frac{d}{dt}\Phi_{\gamma^\pm}(t, s)\xi \neq 0$. Hence $\Phi_{\gamma^\pm}(t, s)\xi$ and $\frac{d}{dt}\Phi_{\gamma^\pm}(t, s)\xi$ are linearly independent, showing

that all trajectories under $\Phi_{\gamma^{\pm}}$ intersect $X^{\pm}$ transversally. For $0 < \varepsilon < \gamma^{+} - \gamma^{-}$ consider the non-empty open cone

$$C_{\varepsilon}^{+} := \left\{ x \in \mathbb{R}^2 : \langle x, \Gamma A x \rangle > (\gamma^{+} - \varepsilon) \| x \|_{\Gamma}^2 \right\}.$$

By the continuous dependence on initial values and parameters, for every sufficiently small ε there exists $T_{\varepsilon} > 0$ such that no trajectory can be contained entirely in C_{ε}^{+} for longer than T_{ε}, formally, $\Phi_{\gamma^{+} - \varepsilon}(t, s)\xi \in \mathbb{R}^2 \setminus C_{\varepsilon}^{+}$ for every $\xi \in C_{\varepsilon}^{+}$ and some t with $s < t \leqslant s + T_{\varepsilon}$. Moreover, $\lim_{\varepsilon \to 0} T_{\varepsilon} = 0$ due to transversality. Choose $\varepsilon_0 > 0$ so small that $T_{\varepsilon} < t_{+} - t_{-}$ for all $\varepsilon \leqslant \varepsilon_0$. Then $[\gamma^{+} - \varepsilon_0, \gamma^{+}] \subset \Sigma_{\Gamma}^{I}(A)$. To see this, pick $\gamma^{+} - \varepsilon_0 < \gamma < \gamma^{+}$ and suppose that γ was not a spectral value. Then the rank of the associated invariant projector P can neither equal two (because $C_{\gamma^{+} - \gamma}^{+} \neq \emptyset$) nor zero (because $\mathbb{R}^2 \setminus C_{\gamma^{+} - \gamma}^{+} \neq \emptyset$); as $\ker P(s) \setminus \{0\}$ would have to be a subset of $C_{\gamma^{+} - \gamma}^{+}$ yet $T_{\gamma^{+} - \gamma} < t_{+} - t_{-}$, it is also impossible to have $\operatorname{rk} P = 1$. A similar argument applies at γ^{-}, and overall, for any $I \in \mathcal{F}$ and $\Gamma = \Gamma^{\top} > 0$ there exists $\varepsilon > 0$ (depending in general on I as well as Γ) such that $[\gamma^{-}, \gamma^{-} + \varepsilon]$ and $[\gamma^{+} - \varepsilon, \gamma^{+}]$ both are subsets of $\Sigma_{\Gamma}^{I}(A)$. The latter therefore is the disjoint union of two non-degenerate intervals. Since $\Sigma_{\Gamma}^{I}(A) \subset [\gamma^{-}, \gamma^{+}] = [-\gamma_0, \gamma_0]$ for all I, the above arguments also show that $\Sigma_{\Gamma}^{\mathbb{R}}(A)$ equals the symmetric non-degenerate interval $[-\gamma_0, \gamma_0]$.

Remark 25. The arguments in Example 24 carry over to higher dimensions. Hence it may be conjectured that diagonalisability is (not only sufficient but) also necessary in Theorem 22(i) and that consequently the infimum in (19) is attained *only* if A is diagonalisable.

4. Concluding examples and remarks

The spectral results of the last section (Theorems 17 and 22 as well as Corollary 18) are analogues of classical results for $\Sigma_{\mathrm{dich}}(A)$, and in this regard they are as satisfactory as could reasonably be expected. It is worth recalling, however, that the usefulness of the classical concepts not least rests upon the associated splitting (5) of the extended phase space into spectral manifolds. The latter are uniquely determined and can neatly be characterised in dynamical terms. Unlike the spectrum, spectral manifolds are problematic to deal with in the finite-time context. Concretely, assume that the numbers $\gamma_1 < \gamma_2$ both belong to $\rho(A)$ yet there is some spectral value between them, i.e. $[\gamma_1, \gamma_2] \cap \Sigma_{\Gamma}^{I}(A) \neq \emptyset$. Then, for every $\xi \in \ker P_{\gamma_1} \cap \operatorname{im} P_{\gamma_2}$ and all $t \geqslant s$, the estimate

$$\| \xi \|_{\Gamma} e^{\gamma_1 (t - s)} \leqslant \left\| \Phi(t, s)\xi \right\|_{\Gamma} \leqslant \| \xi \|_{\Gamma} e^{\gamma_2 (t - s)} \tag{20}$$

provides the natural bounds for the growth of solutions through $\ker P_{\gamma_1} \cap \operatorname{im} P_{\gamma_2}$. Recall that the invariant projector P_{γ} is generally not unique (see Example 4). Without uniqueness, therefore, it is not clear whether (20) is of any use at all in the finite-time setting.

Example 26. Consider (1) with $A(t) \equiv \operatorname{diag}[1, -1]$ and let $\Gamma = \operatorname{id}_{2 \times 2}$. As seen in Example 4(i), for any interval $I \in \mathcal{F}$ there are many invariant projectors P such that (1) is hyperbolic on I, with $\operatorname{rk} P = 1$. For the space X_{η} spanned by the unit vector $\eta \in \mathbb{R}^2$, an explicit calculation yields

$$\underline{\lambda}(X_{\eta}) = \eta_1^2 - \eta_2^2, \qquad \bar{\lambda}(X_{\eta}) = \frac{\eta_1^2 - \tau^2 \eta_2^2}{\eta_1^2 + \tau^2 \eta_2^2},$$

where $\tau = e^{-2(t_+ - t_-)}$. Note that each of the extremal one-dimensional growth rates

$$\underline{\lambda}^{(1)} = \max_{\eta} \underline{\lambda}(X_\eta) = 1, \qquad \bar{\lambda}^{(1)} = \min_{\eta} \bar{\lambda}(X_\eta) = -1$$

is attained for a *unique* space $X_\eta : \underline{\lambda}(X_\eta) = -1$ if and only if $X_\eta = \text{span}\{e_1\}$, and $\bar{\lambda}(X_\eta) = 1$ precisely if $\eta = \pm e_2$. Though not surprising, these observations show that requiring $\ker P$ and $\text{im } P$ to extremise, respectively, lower and upper growth rates may well lead to a unique invariant projector P.

Example 27. Consider (1) with $A(t) \equiv \text{diag}[1, -1, -2]$ and $\Gamma = \text{id}_{3\times3}$. Example 4(ii) provides, for every $I \in \mathcal{F}$, an interval of invariant projectors P with $\text{rk } P = 1$ such that (1) is hyperbolic. For the space X_η spanned by the unit vector $\eta \in \mathbb{R}^3$ one finds

$$\underline{\lambda}(X_\eta) = \eta_1^2 - \eta_2^2 - 2\eta_3^2, \qquad \bar{\lambda}(X_\eta) = \frac{\eta_1^2 - \tau^2\eta_2^2 - 2\tau^3\eta_3^2}{\eta_1^2 + \tau^2\eta_2^2 + \tau^3\eta_3^2},$$

from which it is easy to deduce that $\underline{\lambda}^{(1)} = \max_\eta \underline{\lambda}(X_\eta) = 1$, $\bar{\lambda}^{(1)} = \min_\eta \bar{\lambda}(X_\eta) = -2$. As in the previous example, these extremal (one-dimensional) growth rates are attained uniquely: $\underline{\lambda}(X_\eta) = 1$ if and only if $X_\eta = \text{span}\{e_1\}$, and $\bar{\lambda}(X_\eta) = -2$ precisely if $\eta = \pm e_3$. To study two-dimensional spaces, let $Y_\eta = \eta^\perp$. An elementary computation leads to

$$\underline{\lambda}(Y_\eta) = -\frac{3}{2}\eta_1^2 - \frac{1}{2}\eta_2^2 - \frac{1}{2}\sqrt{\left(2 + 3\eta_1^2 + \eta_2^2\right)^2 - 24\eta_1^2},$$

together with a slightly more involved expression for $\bar{\lambda}(Y_\eta)$, from which the extremal (two-dimensional) growth rates $\underline{\lambda}^{(2)} = \max_\eta \underline{\lambda}(Y_\eta) = -1$ and $\bar{\lambda}^{(2)} = \min_\eta \bar{\lambda}(Y_\eta) = -2$ follow. Moreover,

$$\underline{\lambda}(Y_\eta) = -1 \quad \Longleftrightarrow \quad 3\eta_1^2 \leqslant 2 \quad \text{and} \quad \eta_2 = 0,$$

as well as

$$\bar{\lambda}(Y_\eta) = -2 \quad \Longleftrightarrow \quad \left(2 + \tau^3\right)\eta_1^2 \geqslant 2 \quad \text{and} \quad \eta_2 = 0,$$

and hence the spaces associated with $\underline{\lambda}^{(2)}$, $\bar{\lambda}^{(2)}$ are *not* unique, see also Fig. 1. Note, however, that $\bar{\lambda}^{(2)}$ would be realised uniquely if $t_+ - t_- \to +\infty$.

The somewhat disquieting results of Example 27 have to be qualified in two respects. Firstly, an appropriate choice of Γ may make the problems disappear, at least partly.

Example 28. In the setting and with the notation of Example 27, choose

$$\Gamma = \begin{bmatrix} 4 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A = \operatorname{diag}[1, -1, -2]$$
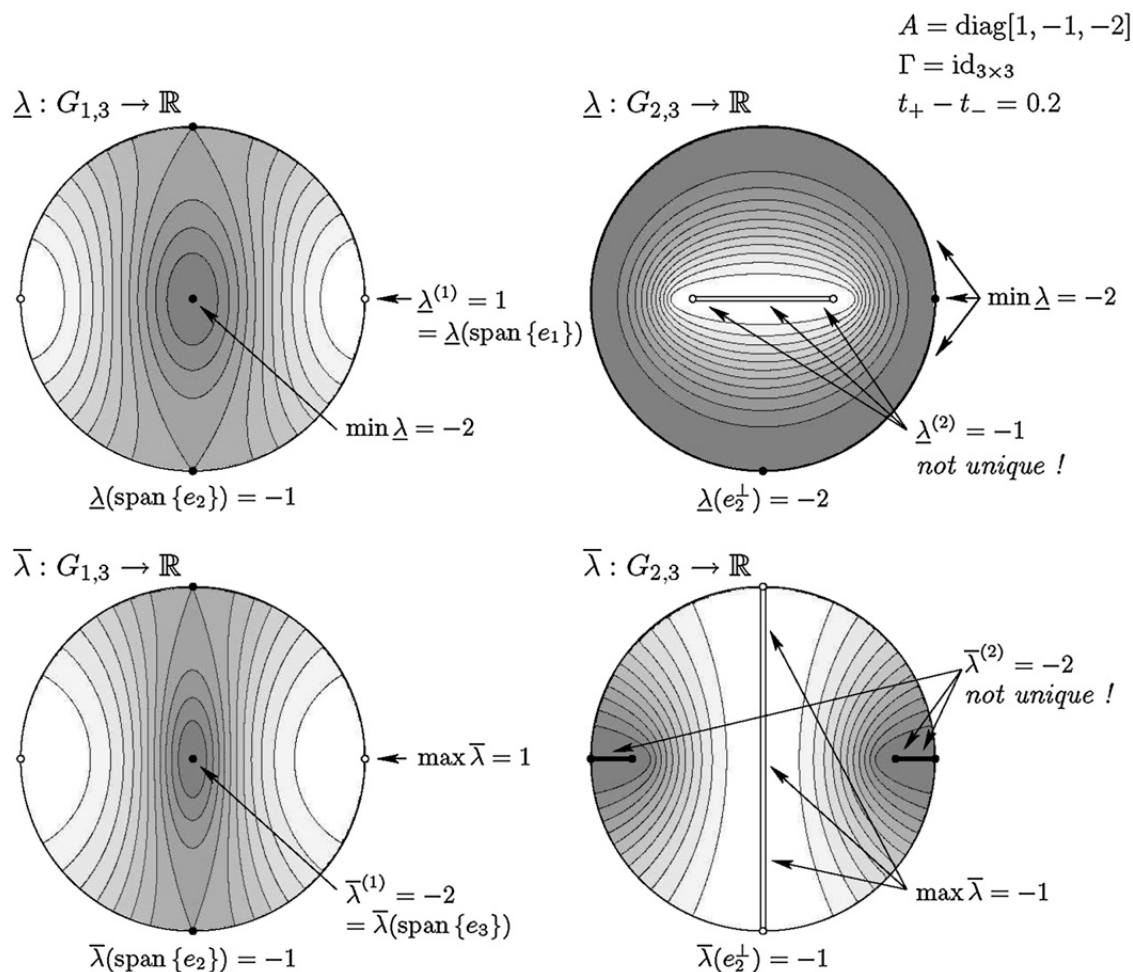$$\Gamma = \operatorname{id}_{3\times3}$$
$$t_+ - t_- = 0.2$$

Fig. 1. Visualising one-dimensional (left) and two-dimensional growth rates by means of their level sets over the Grassmannians $G_{1,3}$ and $G_{2,3}$, respectively, represented as the closed unit disc with centre span$\{e_3\}$ and $e_3^\perp$; any two diametrically opposite points on the periphery (corresponding to $\eta_3 = 0$) have to be identified. While the extremal rates $\underline{\lambda}^{(1)}, \bar{\lambda}^{(1)}$ are attained uniquely, their counterparts $\underline{\lambda}^{(2)}, \bar{\lambda}^{(2)}$ are realised for an entire interval of two-dimensional spaces, see Example 27 for details and notation.

With this, the relevant extremal growth rates $\underline{\lambda}^{(1)} = 2$, $\bar{\lambda}^{(2)} = -2$ are attained for a unique one- and two-dimensional space, respectively. Indeed,

$$\underline{\lambda}(X_\eta) = 4\frac{3\eta_1^2 + 2\sqrt{3}\eta_1\eta_2 + \eta_2^2}{4 + 4\sqrt{3}\eta_1\eta_2 - 3\eta_3^2} - 2,$$

and $\underline{\lambda}(X_\eta) = 2$ if and only if $X_\eta = \operatorname{span}\{\sqrt{3}e_1 - e_2\}$. Similarly, $\bar{\lambda}(Y_\eta) = -2$ precisely if $Y_\eta = \{\sqrt{3}e_1 + \tau e_2\}^\perp$, see also Fig. 2.

In Example 28, Γ is found as a solution of the nonlinear equation that arises from requiring that $\sqrt{\Gamma}A\sqrt{\Gamma}^{-1} + (\sqrt{\Gamma}A\sqrt{\Gamma}^{-1})^\top$ be diagonal with two diagonal entries coinciding. It is not clear whether the analogous nonlinear equations in higher dimensions always has a solution. Even if this turns out not to be the case, however, it is plausible that by choosing an appropriate Γ
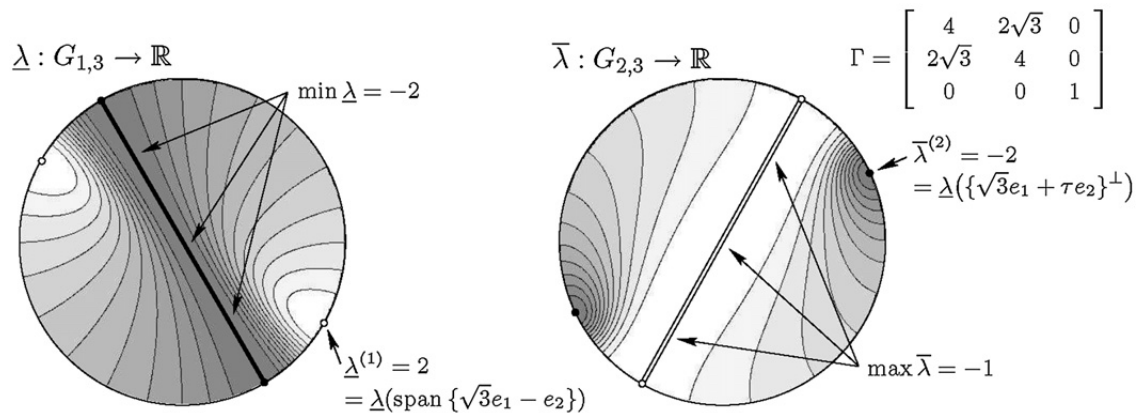
Fig. 2. With the appropriate Γ, the relevant growth rates are realised by a *unique* space, see Example 28.

most non-uniqueness issues can be resolved. For this, it may also be desirable to utilise smooth norms beyond the class considered in this article, i.e. norms not necessarily induced by an inner product.

A second aspect of Example 27 is that it is quite degenerate. While non-uniqueness is in fact robust in the sense that it arises similarly for every $A(t) \equiv \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$ with $\lambda_1 > \lambda_2 > \lambda_3$, it is likely to disappear in more general situations. As the specific computations for Example 27 are unnecessarily involved for the purpose of the present, informal discussion, we explain this point with an even simpler example in $\mathbb{R}^2$.

Example 29. Consider the family of matrices $A_\varepsilon = \mathrm{id}_{2\times 2} + 2\varepsilon J_2(0) = \left[\begin{smallmatrix} 1 & 2\varepsilon \\ 0 & 1 \end{smallmatrix}\right]$, with $\varepsilon \geqslant 0$, and let $\Gamma = \mathrm{id}_{2\times 2}$. If $\varepsilon = 0$ then $\underline{\lambda}(X) = \bar{\lambda}(X) = 1$ for every one-dimensional subspace X, and trivially neither of the extremal one-dimensional growth rates $\underline{\lambda}^{(1)} = \bar{\lambda}^{(1)} = 1$ is attained uniquely. To analyse the case $\varepsilon > 0$, recall from Example 24 that $0 \in \rho(A_\varepsilon)$ whenever $|\varepsilon| < 1$. Assume from now on that $0 < \varepsilon < \min\{1, (t_+ - t_-)^{-1}\}$. A straightforward computation shows that for $X_\varphi = \mathrm{span}\{e_1 \cos\varphi + e_2 \sin\varphi\}$ and $0 \leqslant \varphi \leqslant \pi$

$$\underline{\lambda}(X_\varphi) = \begin{cases} 1 - \varepsilon & \text{if } \frac{3}{4}\pi \leqslant \varphi < \varphi^*, \\ 1 + \varepsilon \min\left\{\sin 2\varphi, \ \frac{\sin 2\varphi + 2\Delta(1 - \cos 2\varphi)}{1 + 2\Delta \sin 2\varphi + 2\Delta^2(1 - \cos 2\varphi)}\right\} & \text{otherwise,} \end{cases}$$

where $\Delta = \varepsilon(t_+ - t_-)$ and $\varphi^* = \frac{1}{2}\pi + \arctan(1 + 2\Delta)$; a similar formula can be derived for $\bar{\lambda}(X_\varphi)$. Note that, for every $\varepsilon > 0$, the growth rate $\underline{\lambda}(X_\varphi)$ becomes maximal for the *unique* space X_{φ_ε} with $\cot\varphi_\varepsilon = \sqrt{\Delta^2 + 1} - \Delta$, see Fig. 3. Interestingly, even the (less relevant) *minimum* of $\underline{\lambda}(X_\varphi)$ is attained only on an interval whose length goes to zero as $\varepsilon \searrow 0$. The non-uniqueness for $\varepsilon = 0$ therefore appears as a very degenerate phenomenon indeed.

Generic uniqueness of spaces realising extremal growth rates becomes plausible also from the following, more general consideration. One simplifying yet non-generic feature of Example 27 is that e.g. $\sup_{\xi \in X\setminus\{0\}} \frac{\mathrm{d}}{\mathrm{d}t} \|\Phi(t, t_-)\xi\| / \|\Phi(t, t_-)\xi\|$ is extremal either for $t = t_-$ or $t = t_+$, *uniformly in* $X \in G_{k,d}$. In general, for a continuous function A such a uniformity is very unlikely to occur. Rather, the growth rates $\underline{\lambda}(X), \bar{\lambda}(X)$ are attained for some $t \in I$ that depends on X. Consequently, $\underline{\lambda}, \bar{\lambda}$ become more generic bounded functions on the compact metric space $G_{k,d}$ and hence typically attain a unique maximum and minimum. Clearly, this plausibility argument is not meant
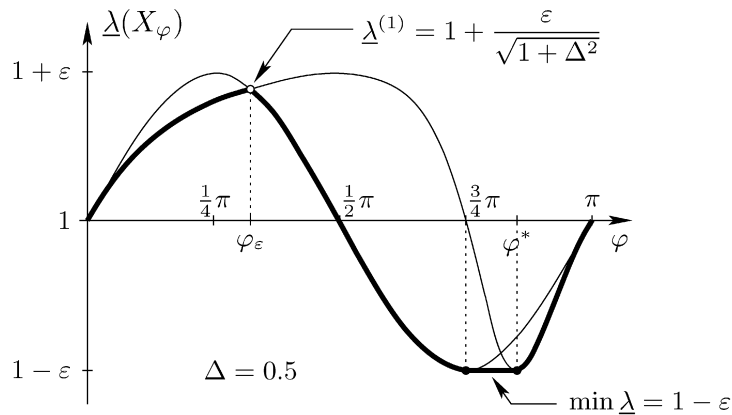
Fig. 3. For every sufficiently small $\varepsilon > 0$ the extremal growth rate $\underline{\lambda}^{(1)}$ is attained for a unique φ_ε; note that $\lim_{\varepsilon \searrow 0} \varphi_\varepsilon = \frac{1}{4}\pi$.

as a substitute for a rigorous analysis. Such an analysis, ideally leading to mild conditions that guarantee the uniqueness of extremal spectral projections, has yet to be provided. The discussion above, however, indicates that the notion of spectrum introduced here can actually be of use for finite-time dynamics. We mention in closing two natural and important questions in this context.

Recent work has established *dynamic partitions* as a versatile tool for finite-time dynamics, see e.g. [2]. Upon the partition of the (extended) phase space into regions of different dynamical behaviour, another notion of hyperbolicity can be founded. How is the latter related to the hyperbolicity of the present article? This questions is interesting because hyperbolicity based on the dynamic partition may be more accessible computationally, while the definition given here is clearly preferable for theoretical considerations. Only partial answers seem to be known at present [3].

Under a practical perspective, it is essential to have efficient tools available for the computation of the spectrum. The examples above indicate that $\Sigma(A)$ can be quite hard to compute directly from its definition, even for very simple A. The finite-time nature of $\Sigma(A)$, however, should make a computational approach feasible. Moreover, any such approach should work for rather large classes of functions $t \mapsto A(t)$ alike and thus markedly contrast the situation for $\Sigma_{\mathrm{dich}}(A)$ whose calculation in general is a notoriously hard problem.

References

[1] B. Aulbach, Gewöhnliche Differentialgleichungen, Spektrum Akademischer Verlag, 2004.
[2] A. Berger, T.S. Doan, S. Siegmund, Nonautonomous finite-time dynamics, Discrete Contin. Dyn. Syst. Ser. B 9 (2008) 463–492.
[3] A. Berger, T.S. Doan, S. Siegmund, Two notions of finite-time hyperbolicity, in: Proceedings of EQUADIFF 2007, in press.
[4] M. Berger, B. Gostiaux, Differential Geometry: Manifolds, Curves, and Surfaces, Springer, 1988.
[5] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math., vol. 629, Springer, Berlin, New York, 1978.
[6] W. Hahn, Stability of Motion, Springer, 1967.
[7] G. Sell, The Floquet Problem for Almost Periodic Linear Differential Equations, Lecture Notes in Math., vol. 415, 1974, pp. 239–251.
[8] S. Siegmund, Dichotomy spectrum for nonautonomous differential equations, J. Dynam. Differential Equations 14 (2002) 243–258.